Zeta functions (or, how to count pts.!)

- Let $k = \mathbb{F}_q$ for some prime-power q.
- Let C be a smooth projective curve defined over k.
- We keep thinking of C as maxSpec S, where S is the (graded) coordinate ring of C.
- Also, $C \cong C_K$ where K is the function field of C.

- Our interest now is in point-counting. I.e. for $n \in \mathbb{N}_{\geq 1}$, how well can we estimate $N_n :=$ number of $\mathbb{F}_{q^n}$-points in C.

▷ Equivalently, $N_n = \#\{P \in C \mid d(P) \mid n\}$.
- Eg. $N_1 =$ number of k-points in C.

- Why not study its generating function $G(t) := \sum_{n \geq 1} N_n \cdot t^n \in \mathbb{Z}[[t]]$?

the ring of univariate power series over integers

- With $G(t)$ as our goal, we define the following power series (which will have better structure):

- **Defn**: The <u>zeta function of C</u> (over k) is
$$\underline{Z(t)} := \sum_{\substack{D \geq 0 \\ \text{in } \mathrm{Div}(C)}} t^{d(D)} \in \mathbb{Z}[[t]].$$

- Recall that $d(\cdot)$ does not change up to divisor class.
- Also, we have an exact sequence:
$$0 \to Cl_0(C) \xrightarrow{\subseteq} Cl(C) \xrightarrow{d(\cdot)} \mathbb{Z}$$

- We do not know (yet!) the image of $d(\cdot)$. But, since it is an ideal of $\mathbb{Z}$, it has to be of the form $\underline{\delta\mathbb{Z}}$, for some $\delta \in \mathbb{N}$.

▷ $0 \to Cl_0(C) \xrightarrow{\subseteq} Cl(C) \xrightarrow{d(\cdot)} \delta\mathbb{Z} \to 0$ is an exact sequence.

- Define $\underline{Cl_d(C)}$ to be the set of divisor classes of deg $= d$. (We only consider $d \in \delta\mathbb{Z}$.)

- Now we can start expanding the zeta fn.

$$Z(t) = \sum_{D \geq 0} t^{d(D)} = \sum_{d \in \delta\mathbb{N}} \sum_{\mathcal{D} \in Cl_d(C)} \sum_{\substack{D \in \mathcal{D} \\ D \geq 0}} t^d .$$

- We try to understand the inner two sums:

- Lemma 1: Let $\mathcal{D} \in Cl_d(C)$ for $d \in \mathbb{N}$. Then,

$$\#\{D \in \mathcal{D} \mid D \geq 0\} = (q^{\ell(\mathcal{D})} - 1)/(q-1).$$

Pf: • For any $D, D' \in \mathcal{D}$ there is $f \in K^*$ s.t. $D' - D = (f)$.

$\Rightarrow \quad D + (f) = D' \geq 0$, if D' is positive.

• Moreover, for fns. $f_1, f_2 \in K^*$, $(f_1) = (f_2)$ iff $\exists c \in k^*$ s.t. $f_1 = c \cdot f_2$.

• Thus, if we fix a $D \in \mathcal{D}$, then the number of positive divisors in $\mathcal{D}$ equals the number of nonsimilar fns. in $L(D)$.

• The latter equals $(q^{\ell(\mathcal{D})}-1)/(q-1)$. $\square$

- <u>Lemma 2</u>: $\forall d \in \delta\mathbb{Z}$, $|Cl_d(C)| = |Cl_0(C)| < \infty$.

<u>Pf</u>: • First, consider $d \in \delta\mathbb{Z}$ s.t. $d > (g-1)$.
• Let $D \in \mathcal{D} \in Cl_d(C)$. By Riemann's theorem:
$\ell(D) \geqslant d(D) + 1 - g > 0$.
$\Rightarrow \exists$ a positive divisor in $\mathcal{D}$.
• Thus, $|Cl_d(C)|$ is upper-bounded by the number of <u>non-equivalent</u> positive divisors of deg d.
• Notice that if a point $P \in C$ appears in such a divisor D, then:
$d(P) \leq d(D) \leq d$.
• But, the number of such P's is <u>finite</u>. (Why?)
$\Rightarrow |Cl_d(C)| < \infty$.
• Finally, notice that for $d, d' \in \delta\mathbb{Z}$, and divisor classes $\mathcal{D} \in Cl_d(C)$, $\mathcal{D}' \in Cl_{d'}(C)$ we have the bijection

$$Cl_d(C) \longrightarrow Cl_{d'}(C)$$
$$\mathcal{E} \longmapsto \mathcal{E} + \mathcal{D}' - \mathcal{D}$$

- Using this bijection we get the result, $\square$

- Defn: We call $|Cl_0(C)|$ the <u>class number of C</u> (over k), denoted by $\underline{h(C)}$.

- So, $Z(t) = \sum_{d \in \delta\mathbb{N}} \sum_{\mathcal{D} \in Cl_d(C)} \sum_{\substack{D \in \mathcal{D} \\ D \geq 0}} t^d$

$$= \sum_{\substack{d \in \delta\mathbb{N} \\ \mathcal{D} \in Cl_d(C)}} t^d \cdot \frac{q^{\ell(\mathcal{D})} - 1}{q - 1}$$

- We know that for a positive $D \in \mathcal{D}$:

$$\ell(D) - d(D) \leq \ell(0) - d(0) = 1.$$

$$\Rightarrow \ell(\mathcal{D}) = \ell(D) \leq d(D) + 1 = (d+1).$$

$$\Rightarrow Z(t) \leq \sum_{d \geq 0} h(C) \cdot (qt)^d \cdot (d+1).$$ Thus,

- Proposition: $Z(t)$ converges for $t \in \mathbb{C}$, if $|t| < q^{-1}$.
  - Also, $\underline{Z(q^{-s})}$ converges for $s \in \mathbb{C}$, if $Re(s) > 1$.

This lets us view the <u>power series</u> Z also as a <u>complex fn.</u>!