Theorem (Riemann-Roch, 1865): Let $D \in \mathrm{Div}(C)$ and W be a divisor in the canonical class of K. Then, $\ell(D) - d(D) = 1 - g + \ell(W-D)$.

Proof: • Let $\omega \in \Omega(W)$ and consider the map

$\varphi : L(W-D) \longrightarrow \Omega(D)$; $x \mapsto x\omega$

• We know that φ is a k-lin. injection.

• Let $\omega' \in \Omega(D)$. Since, $\dim_K \Omega = 1$, $\exists\, x \in K^*$ s.t. $\omega' = x\omega$.

$\Rightarrow$ In $Cl(C)$: $(x) + W = (x) + (\omega) = (x\omega) = (\omega')$

$\geqslant D$. [$\because (\omega')$ is maximal.]

$\Rightarrow \quad (x) \geqslant D - W \Rightarrow x \in L(W-D)$.

$\Rightarrow \varphi$ is a k-lin. isomorphism.

$\Rightarrow \quad \ell(W-D) = \delta(D)$. $\square$

- **Corollary 1:** $\ell(W) = g$ & $d(W) = 2g-2$.

Pf: • In Riemann-Roch, $D=0$ gives $1 - 0 = 1 - g + \ell(W)$.

• While, $D = W$ gives $g - d(W) = 1 - g + 1$. $\square$

Jacobian variety

- Let D be a divisor with $d(D) > d(W) = 2g-2$. Then, $d(W-D) < 0$. Thus, $L(W-D) = \{0\}$, implying $\ell(W-D) = 0$.

- Corollary 2: $d(D) > 2g-2 \Rightarrow \ell(D) - d(D) = 1-g$.

- Thus, $d(D) > 2g-2 \geq 0 \Rightarrow \ell(D) \doteq 2g-1+1-g = g > 0$.

- In other words, for such D, $\exists x \in K^*$ s.t. $D + (x) \geq 0$. So,

Corollary 3: $d(D) > 2g-2 \geq 0 \Rightarrow \exists D' \geq 0,\ D' \sim D$ (in $Cl(C)$).

- In particular, for such D, there are at most $d(D)$ points $\{P_i \in C \mid i\}$ s.t. $D \sim \sum_i P_i$!

- Rmk: So, every divisor D has a concise, positive representation in the divisor class.

Symmetric Power of C

- Define $\underline{C^{(g)}} := C^g / \mathrm{Symm}_g$, i.e. take the g-fold Segre product, and identify each "point" $(P_1, \ldots, P_g)$ with $(P_{\sigma(1)}, \ldots, P_{\sigma(g)})$, $\forall \sigma \in \mathrm{Symm}_g$

- $C^{(g)}$ is again a smooth projective variety, but, of dimension g.

- We will sketch how Riemann-Roch gives a variety, birational to $C^{(g)}$, which has an abelian group structure.

This is called the Jacobian variety of C, $J(C)$.

- Idea: • Fix a point $P_0 \in C$ of degree one & let $r := 2g-1$.

• For every $D_\gamma \geq 0$ of deg $(g-1)$ we identify a subset of $\mathrm{Div}_0(C)$,

$$\underline{\mathcal{D}_\gamma} := \{ D \in \mathrm{Div}_0(C) \mid \ell(D + rP_0 - D_\gamma) = 1 \}$$

and the corresponding subset of $C^{(g)}$,

$$\underline{C_\gamma} := \{(P_1,\dots,P_g)\in C^{(g)} \mid D\in\mathcal{D}_\gamma,\ D+rP_0-D_\gamma=\sum_{i=1}^{g}P_i\}.$$

- It can be shown that:

▷ Each C_γ is a quasi-PV in $C^{(g)}$.

▷ By "glueing" these open patches $\{C_\gamma \mid \gamma\}$ we get the Jacobian variety $J(C)$.

▷ Since, each point in $\underline{Div_0(C)}$ corresponds to $\underline{\text{some point in } J(C)}$. The latter inherits the abelian group structure.

▷ When $g=1$, i.e. elliptic curves C, $J(C)\cong C$. Hence, C is an abelian group!