$\triangleright$ $D \leq D'$ in $Div(C) \Rightarrow \Omega_{K/k}(D') \subseteq \Omega_{K/k}(D)$.

- The differentials of K we denote by $\underline{\Omega_{K/k}} := \bigcup_D \Omega_{K/k}(D)$.

- **Proposition:** (i) $\Omega_{K/k}$ is a K-vector space.

(ii) $\dim_K \Omega_{K/k} = 1$.

Proof: (i) • Let $w \in \Omega(D)$ and $x \in K$. Then, we can define $\underline{x \cdot w}$ to mean the map:
$\mathbb{A}/(\mathbb{A}(D)+K) \to k$; $r \mapsto w(r \cdot x)$.

• Let $w' \in \Omega(D')$. Let E be a divisor s.t. $E \leq D$ & $E \leq D'$. Then, we can define $\underline{(w+w')}$ as the map:
$\mathbb{A}/(\mathbb{A}(E)+K) \to k$; $r \mapsto w(r)+w'(r)$.

• These definitions make Ω a K-vector space.

Note: $xw \in \Omega(D+(x))$ & $w+w' \in \Omega(\gcd(D,D'))$.

(ii) • Let $w \in \Omega(D)$ & $w' \in \Omega(D')$ be two nonzero differentials. We will show them K-lr. dependent.

• Pick a <u>positive</u> divisor E. (So, $\ell(E) \leq d(E)+1$.)

• Consider the two k-lin. homomorphisms:

$$\begin{array}{ll} L(D+E) \xrightarrow{\ i\ } \Omega(-E)\ ; & x \mapsto x\omega \\ L(D'+E) \xrightarrow{\ i'\ } \Omega(-E) & x' \mapsto x'\omega' \end{array}$$

• <u>Claim</u>: i & i' are injective homomorphisms.

<u>Pf</u>: • $x\omega \in \Omega(D+(x))$

• $\because D+(x) \geqslant D-(D+E) = -E$, we deduce
$x\omega \in \Omega(D+(x)) \subseteq \Omega(-E)$.

• If $x\omega = 0$ in $\Omega(D+(x))$, then the map $x^{-1}(x\omega) = 0$ in $\Omega(D+(x)+(x^{-1}))$, or
$\omega = 0$ in $\Omega(D)$.

Thus, a nonzero ω yields a <u>nonzero</u> $x\omega \in \Omega(-E)$.

• Thus, $\dim_k(\operatorname{img}(i)) + \dim_k(\operatorname{img}(i'))$

$= \ell(D+E) + \ell(D'+E)$

$\geqslant d(D+E)+1-g + d(D'+E)+1-g$ [by Riemann thm.]

$= 2\cdot d(E) + d(D+D') + 2(1-g)$

$> d(E)+g-1 = \ell(-E) - d(-E) + g - 1 = \dim_k \Omega(-E)$.

[pick E <u>large</u> enough] [this is 0] [Proposition 2]

- This means that $\operatorname{img}(i) \cap \operatorname{img}(i') \neq \{0\}$.

$\Rightarrow \exists\, x, x' \in K^*$ st. $x\omega = x'\omega'$.

$\Rightarrow \dim_K \Omega_{K/k} = 1$. $\square$

- Could we <u>associate a divisor</u> to this 1-dim. k-vec. space $\Omega_{K/k}$? If we can, then we might realize $\delta(D)$ as an $\ell(\cdot)$!

<u>Proposition</u>: (i) For any nonzero $\omega \in \Omega_{K/k}$, the set <u>$M(\omega)$</u> $:= \{D \in \operatorname{Div}(C) \mid \omega \in \Omega(D)\}$ has

(wrt $\geqslant$) a <u>unique</u> maximal element, denoted by <u>(ω)</u>.

(ii) $\forall x \in K^*, \omega \in \Omega_{K/k}$: $(x\omega) = (x) + (\omega)$.

<u>Proof</u>: (i) First, note that for every $D \in M(\omega)$, $d(D)$ is bounded:

Consider the map $L(D) \to \Omega(0)$; $x \mapsto x\omega$. Since it is a k-li. injection, we deduce that $\ell(D) \leq \delta(0) = \ell(0) - d(0) + g - 1 = g$.

Also, $\ell(D) - d(D) = \delta(D) + 1 - g \geqslant 1 - g$.

$\Rightarrow d(D) \leqslant 2g - 1$.

- Thus, we can pick a divisor $D_\omega \in M(\omega)$ of maximal degree. Say, D'_ω is another such divisor in $M(\omega)$.

Then, consider $\Omega(\mathrm{lcm}(D_\omega, D'_\omega))$.

Exercise: Show that it is $\Omega(D_\omega) \cap \Omega(D'_\omega)$.

- Thus, $\mathrm{lcm}(D_\omega, D'_\omega) \in M(\omega)$.
- By the maximality we then deduce $D_\omega = D'_\omega$.
- Thus, D_ω is unique, denoted by (ω). $\square$

(ii) • Since, $\omega \in \Omega((\omega))$ we have $x\omega \in \Omega((\omega)+(x))$

- Further, $x\omega \in \Omega(D) \Leftrightarrow \omega \in \Omega(D-(x))$.
- This, together with $d((x))=0$, means that the maximal element in $M(x\omega)$ is $(\omega)+(x)$.

$\Rightarrow (x\omega) = (x) + (\omega)$. $\square$

- We now know that every nonzero differential has the same divisor class!

Defn: The img of $\Omega_{K/k} \to \mathrm{Div}(C) \to \mathrm{Cl}(C)$ is the canonical class of K.

$$\omega \mapsto (\omega) \mapsto (\omega) \bmod \mathrm{Div}_a(C)$$