(iii) • Firstly, $\dim_k A/(A(D)+k) \leq \delta(D)$:

Suppose there are $h$ elements $r_1, \dots, r_h \in A$ which are $k$-lr. indep. modulo $(A(D)+k)$.

• By (i), we get divisors $D_{r_1}, \dots, D_{r_h}$ (positive) s.t. $\forall i \in [h]$, $r_i \in A(D_{r_i})$.

• By taking $D'$ to be the $\operatorname{lcm}\{D_{r_i} \mid i\}$, we deduce $\forall i \in [h]$, $r_i \in A(D')$.

$\Rightarrow r_1, \dots, r_h$ are $k$-lr. indep. in $(A(D')+k)/(A(D)+k)$.

• Now, $(A(D')+k)/(A(D)+k)$

For vec. spaces: $U+V/V \cong U/U \cap V$.

$$\cong A(D')/(A(D') \cap (A(D)+k))$$
$$\cong A(D')/(L(D')+A(D))$$
$$\cong (A(D')/A(D))/((L(D')+A(D))/A(D))$$
$$\cong (A(D')/A(D))/(L(D')/(L(D') \cap A(D)))$$
$$\cong (A(D')/A(D))/(L(D')/L(D))$$

which has dimension (over $k$) $=$

$$d(D') - d(D) - (\ell(D') - \ell(D))$$
$$= (\ell(D) - d(D)) - (\ell(D') - d(D'))$$
$$\leq \ell(D) - d(D) - (1-g) = \delta(D).$$

- Next we show, $\dim_k \mathbb{A}/(\mathbb{A}(D)+K) \geq \delta(D)$.
- By Riemann theorem, $\exists D_0$ s.t.
  $\ell(D_0) - d(D_0) = 1-g$.
- Consider $D' := \mathrm{lcm}(D_0, D)$. Then, since $D' \geq D_0$, we have $\ell(D') - d(D') = 1-g$.
- From the previous calculation, we have:
  $\dim_k (\mathbb{A}(D')+K)/(\mathbb{A}(D)+K) =$
  $$(\ell(D)-d(D)) - (\ell(D')-d(D')) = \delta(D).$$
  $\Rightarrow \dim_k \mathbb{A}/(\mathbb{A}(D)+K) \geq \delta(D)$. $\square$

## <u>Differentials</u>

- We want a re-interpretation of $\delta(D)$ as the $\ell(\cdot)$ of some divisor. A key tool there is differentials.

- <u>Defn</u>: • A (pseudo-) <u>differential of $K$</u> is an $\omega \in \mathrm{Hom}_k(\mathbb{A}/(\mathbb{A}(D)+K), k)$, for some divisor $D$.
  - Denote $\underline{\Omega_{K/k}(D)} := \mathrm{Hom}_k(\mathbb{A}/(\mathbb{A}(D)+K), k)$.

▷ $D \leq D'$ in $Div(C) \Rightarrow \Omega_{K/k}(D') \subseteq \Omega_{K/k}(D)$.

- The differentials of $K$ we denote by
$\underline{\Omega_{K/k}} := \bigcup_D \Omega_{K/k}(D)$.

- Proposition: (i) $\Omega_{K/k}$ is a $K$-vector space.
(ii) $\dim_K \Omega_{K/k} = 1$.

Proof: (i) • Let $w \in \Omega(D)$ and $x \in K$. Then, we can define $\underline{x \cdot w}$ to mean the map: $\mathbb{A}/(\mathbb{A}(D)+K) \rightarrow k$ ; $r \mapsto w(r \cdot x)$.

• Let $w' \in \Omega(D')$. Let $E$ be a divisor s.t. $E \leq D$ & $E \leq D'$. Then, we can define $\underline{(w+w')}$ as the map:
$\mathbb{A}/(\mathbb{A}(E)+K) \longrightarrow k$; $r \mapsto w(r) + w'(r)$.

• These definitions make $\Omega$ a $K$-vector space.

Note: $xw \in \Omega(D+(x))$ & $w+w' \in \Omega(\gcd(D, D'))$.

(ii) • Let $w \in \Omega(D)$ & $w' \in \Omega(D')$ be two nonzero differentials. We will show them $K$-lr. dependent.