Adeles (technically, "pre"-adeles)

- The idea is to see $\delta(D)$ as a kind of "dual"! Should we study the k-vec. spc. $K/L(D)$? We will realize this by defining adeles:

- Definition: • For the curve C, a tuple $r = (r_P \mid P \in C)$ of rational fns. is called an adele if $v_P(r_P) \geq 0$ for all, except finitely many, $P \in C$.
  • r_P is called the P-th component of r.
  • Denote the set of all adeles, and $\bar{0}$, by $\mathbb{A}$.

- Note: The construction of the adele set $\mathbb{A}$ from K is inspired from that of $\mathbb{R}$ from $\mathbb{Q}$.

- Proposition 1: (i) $\exists$ natural embedding $K \hookrightarrow \mathbb{A}$.
  (ii) $\mathbb{A}$ is a K-algebra (not a field).
  (iii) $\forall P \in C$, $v_P(\cdot)$ extends to $\mathbb{A}$ as well.
  (iv) $\exists$ an analog of $L(D)$ in $\mathbb{A}$, called $\mathbb{A}(D)$.

- Pf: (i) • Map any element $x \in K$ to the tuple $r := (x \mid P \in C)$. We know that $v_P(x) = 0$, for all, but finitely many, $P \in C$.

• Thus, clearly $r \in \mathbb{A}$. From now on we will denote this element r also by x.

(ii) • Let $r = (r_P)$ & $r' = (r'_P)$ be in $\mathbb{A}$. Define $r + r' := (r_P + r'_P)$ & $rr' := (r_P r'_P)$.

• By the valuation axioms, clearly, $r + r'$ & $rr' \in \mathbb{A}$.

• Also, $0, 1 \in \mathbb{A}$. (Zerodivisors eg. $(1, 0, \ldots) \cdot (0, 1, \ldots) = 0$.)

• Thus, $\mathbb{A}$ is a K-algebra.

(iii) • For $P \in C$, we extend the valuation v_P to $\mathbb{A}$ naturally, $\underline{v_P} : \mathbb{A} \to \mathbb{Z}$; $r \mapsto v_P(r_P)$.

• Clearly, $v_P(\cdot)$ satisfies the valuation axioms. (Except, now the domain of v_P is <u>not</u> a field!)

(iv) • Let $D \in \mathrm{Div}(C)$. For adeles $r, s \in \mathbb{A}$ we write $r \equiv s \pmod D$ if $\forall P \in C$, $v_P(r-s) \geq \mathrm{ord}_P(D)$.

▷ $\equiv_D$ is an equivalence relation on $\mathbb{A}$.

• Similarly, $\underline{\mathbb{A}(D)}$ is defined as the set: $\{r \in \mathbb{A} \mid \forall P \in C,\ v_P(r) \geq -\mathrm{ord}_P(D)\}$.

• Like $L(D)$, $\mathbb{A}(D)$ is also a k-vector space. □

— We now investigate adeles vis-a-vis divisors.

- <u>Proposition 2</u>: (i) Adele r has an associated positive divisor D_r.

(ii) Divisors $D' \geq D \Rightarrow \dim_k \mathbb{A}(D')/\mathbb{A}(D) = d(D') - d(D)$.

(iii) $\dim_k \mathbb{A}/(\mathbb{A}(D)+K) = \delta(D)$.

<u>Proof</u>:

(i) Define $\underline{D_r} := \sum\{-v_P(r)P \mid P \in C,\ v_P(r) < 0\}$.
It is a divisor because such P are finite.
Moreover, $r \in \mathbb{A}(D_r)$.

(ii) • Let $S := \mathrm{supp}(D') \cup \mathrm{supp}(D)$. It suffices to show an <u>isomorphism</u>: $L(D')_S / L(D)_S \to \mathbb{A}(D')/\mathbb{A}(D)$.
[$\because$ the $\dim_k$ of LHS, we know, is $d(D'-D)$.]

- Start with the map $\phi: L(D)_S \to \mathbb{A}(D);\ x \mapsto r_x$. Where, for all $P \in C$,
$$(r_x)_P := \begin{cases} x, & \text{if } P \in S \\ 0, & \text{else.} \end{cases}$$
- Clearly, ϕ is k-linear homomorphism.
- It easily extends to an injection:
$$\phi: L(D')_S / L(D)_S \longrightarrow \mathbb{A}(D') / \mathbb{A}(D).$$
- <u>Claim</u>: The ϕ above is onto.

<u>Pf</u>: • Let $r \in \mathbb{A}(D')$. By the approx. thm. find a $u \in K^*$ s.t. $v_P(u - r_P) \geq -\mathrm{ord}_P D$, $\forall P \in S$. (Note: S is finite.)
$$\Rightarrow v_P(u) \geq \min\{v_P(r_P), -\mathrm{ord}_P(D)\}$$
$$\geq \min\{-\mathrm{ord}_P(D'), -\mathrm{ord}_P(D)\}.$$
$$\Rightarrow u \in L(D')_S.$$
- Consider the image $\phi(u) =: r_u$. We will show that $r_u - r \in \mathbb{A}(D)$:
- $\forall P \in S$, $(r_u)_P - r_P = u - r_P$ which has $v_P(\cdot) \geq -\mathrm{ord}_P(D)$.
- $\forall P \notin S$, $(r_u)_P - r_P = -r_P$ of $v_P(\cdot) \geq -\mathrm{ord}_P(D') = 0$. $\square$