The Riemann theorem (1857)

<u>Theorem</u>: There exists a $g = g(C,k) \in \mathbb{N}_{\geq 0}$ s.t. $\forall x \in K \setminus k$, $1-g = \min_{m \in \mathbb{Z}} \{ \ell((x^m)_\infty) - d((x^m)_\infty) \}$.

We call g the <u>genus of K</u>. Further, for any $D \in \mathrm{Div}(C)$, $\ell(D) - d(D) \geq 1-g$.

- <u>Pf</u>: • Let us begin with an $x \in K \setminus k$. Let $N := [K : k(x)]$ & large enough $s \in \mathbb{N}_{\geq 0}$.

• We know from the previous proof, that: for all $t \in \mathbb{N}$, $Nt \leq \ell((x^{s+t})_\infty)$.

• Writing $m = s+t$, we deduce that $\forall m \geq s$, $\ell((x^m)_\infty) - d((x^m)_\infty) \geq Nt - m \cdot N = -sN$.

• Thus, the integer $\underline{\underline{\mu_x}} := \min_{m \in \mathbb{N}} \{ \ell((x^m)_\infty) - d((x^m)_\infty) \}$ actually exist!

• Now consider a $D \in \mathrm{Div}(C)$. Express it as a difference $D = D_0 - D_\infty$ of <u>positive</u> divisors.

- From $D_0 \geq D$, we deduce:

$$\ell(D) - d(D) \geq \ell(D_0) - d(D_0).$$

- Note that both $\ell(\cdot)$ & $d(\cdot)$ are <u>constant on a divisor class</u>. (Say, $z \in K^*$. Then, it is easily seen that $d(D+(z)) = d(D)$. Further, $L(D) \longrightarrow L(D+(z))$; $f \mapsto z^{-1}f$ is a k-isomorphism. So $\ell(D+(z)) = \ell(D)$.)
- Thus, together with $(x^m)_\infty \geq -D_0 + (x^m)_\infty$, we deduce:

$$\ell(-D_0 + (x^m)_\infty) - d(-D_0 + (x^m)_\infty) \geq \ell((x^m)_\infty) - d((x^m)_\infty) \geq \mu_x.$$

$\Rightarrow \ell(-D_0 + (x^m)_\infty) \geq -d(D_0) + mN + \mu_x > 0$, for large enough m.

$\Rightarrow z \in L(-D_0 + (x^m)_\infty)$.

$\Rightarrow (x^m)_\infty \geq D_0 - (z)$.

$\Rightarrow \ell(D_0) - d(D_0) \geq \ell((x^m)_\infty) - d((x^m)_\infty) \geq \mu_x$.

$\Rightarrow \ell(D) - d(D) \geq \mu_x$.

- In particular, it means that μ_x is independent of x! $\square$

g of K

- The genus depends on k. When k is alg. closed then g is called the <u>arithmetic genus of the curve C</u>. For $k=\mathbb{C}$, it coincides with the <u>topological notion of genus</u> (classical).

- <u>Vague remarks</u> about <u>curve genus in $\mathbb{P}^2_{\mathbb{C}}$</u> :
  - Genus of a proj. curve $f(x,y,z)=0$ of deg $d=3$ is <u>the same</u> as that of $h = l_1 \cdot l_2 \cdot l_3$ where l_1, l_2, l_3 are linear forms in $\mathbb{C}[x,y,z]$.
  - As each l_i is a complex proj. line, we can "draw" it as the surface of a <u>real sphere</u>. Further, l_i, l_j $(i \neq j)$ have exactly <u>one point</u> in common.
  - So, we get the following "real" diagram for the proj. complex curve h :

The real surface realizing h

▷ There is a <u>loop</u> in this surface that cannot be continuously shrunk to a point!

$\Rightarrow$ genus $= 1$.

▷ This suggests that a proj. complex curve of degree d should have topological genus $\leqslant$ $\binom{d}{2} - (d-1) = \binom{d-1}{2}$.

[$(d-1)$ common pts., in the above picture, are utilized to form a connected chain of the d spheres; the rest give us loops!]

– Continuing with the algebraic genus:

– <u>Corollary</u>: $g \geqslant 0$.

<u>Pf</u>: By Riemann theorem, we have for $D=0$: $1 = \ell(0) - d(0) \geqslant 1-g$.

$\Rightarrow g \geqslant 0$. $\square$

– <u>Defn</u>: The <u>degree of speciality</u> of a divisor D is defined as $\underline{\delta(D)} := \ell(D) - d(D) + g - 1$.

– How is $\delta(D)$ related to D?

This was answered in an exact way by Roch (Riemann's student)!