- **Theorem:** $\forall x \in K \setminus k$, $d((x)_0) = d((x)_\infty) = [K : k(x)]$.

Pf: • Let $N := [K : k(x)]$, $D := (x)_0$ with support S.

• First, we show $d(D) \leq N$:

• Let $y_0, \dots, y_N \in L(0)_S$ be rational functions.

• Then, by the defn. of N, $\exists f_0, \dots, f_N \in k[x]$ not all zero, st. $\sum_{j=0}^{N} f_j y_j = 0$.

• Further, we can assume that not all the elements $a_j := f_j(0)$, $0 \leq j \leq N$, are zero.

• On rewriting we get, $\sum_{j=0}^{N} a_j y_j = -x \cdot \sum_{j=0}^{N} g_j y_j$.

• $\forall p \in S$, we have $v_p(\sum a_j y_j) = v_p(x) + v_p(\sum g_j y_j)$

$\geqslant v_p(x) + \min_j \{v_p(g_j) + v_p(y_j)\} \geqslant v_p(x) = \mathrm{ord}_p(D)$.

$\Rightarrow \sum a_j y_j \in L(-D)_S$.

$\Rightarrow \dim_k L(0)_S / L(-D)_S \leq N$.

• Since we already know $LHS = d(0) - d(-D) = d(D)$, we get $d(D) \leq N$.

- Now, we show $d(D) \geq N$:
- Let $y_1, \dots, y_N$ be a $k(x)$-basis of K s.t. they are <u>integral</u> over $k(x)$. [exist?]
- Then, $\{x^i y_j \mid i \in [t], j \in [N]\} =: B$ are k-linear independent, for any $t \in \mathbb{N}$.
- Prove that: If y is integral over $k(x)$ & $v_p(y) < 0$, then $v_p(x) < 0$. [Say, $y^m = f_{m-1}(x) y^{m-1} + \dots + f_0(x)$, for some $f_i \in k[x]$. Taking $v_p(\cdot)$ both sides, we can deduce $v_p(x) < 0$.]
- Thus, for large enough $s \in \mathbb{N}$, the divisors $\{(x^{s+t})_\infty + (x^i y_j) \mid i \in [t], j \in [N]\}$ are all positive.

$\Rightarrow$ B are k-lr. indep. fns. in $L((x^{s+t})_\infty)$.
- This, together with $(x^{s+t})_\infty \geq (x)_\infty$, means:

$|B| = Nt \leq \dim_k L((x^{s+t})_\infty) \leq d((x^{s+t})_\infty) + \ell((x)_\infty) - d((x)_\infty)$.

$\Rightarrow Nt \leq (s+t-1) \cdot d((x)_\infty) + \ell((x)_\infty)$.
- Tending $t \to \infty$, we deduce

$$d((x)_\infty) \geq N.$$

- We can repeat the proof for x^{-1}, & using $k(x)=k(x^{-1})$, we get the inequality:
$$d(D) \geq N. \qquad \square$$

- Corollary: (1) For $x \in K^*$, $d((x)) = 0$.
(2) $0 \xrightarrow{\subseteq} \mathrm{Div}_a(C) \xrightarrow{\subseteq} \mathrm{Div}_0(C) \xrightarrow{\subseteq} \mathrm{Div}(C) \xrightarrow{d} \mathbb{Z}$ is a sequence of (abelian gp.) homomorphisms.

- The above sequence allows us to define:
- Definition: • The group of divisor classes
$$\underline{Cl(C)} := \mathrm{Div}(C)/\mathrm{Div}_a(C).$$
• The group of divisor classes of degree 0
$$\underline{Cl_0(C)} := \mathrm{Div}_0(C)/\mathrm{Div}_a(C).$$

▷ $0 \xrightarrow{\subseteq} Cl_0(C) \xrightarrow{\subseteq} Cl(C) \xrightarrow{d} \mathbb{Z}$ is an exact sequence.
(I.e. image of a map is the kernel of the subsequent map!)
(Is d onto?)