- Interestingly, we could even "measure" how "big" is $L(D')$ compared to $L(D)$ when $D' \geq D$.

- <u>Theorem</u>: Let $D'_S \geq D_S$ in $Div(C)$ for finite $S \subset C$. Then,

$$\dim_k L(D')_S / L(D)_S = d(D'_S) - d(D_S).$$

<u>Pf</u>: - <u>Idea</u>: Induction on $\sum_{P \in S} ord_P(D - D')$. The finiteness of S is required to apply the <u>approximation theorem</u> (on valuations).

- For a $Q \in S$, it suffices to show that $\dim L(D+Q)_S / L(D)_S = d(Q)$.

Because, if $D'_S - D_S = \sum_{i=1}^{h} Q_i$ then we have the chain:

$$L(D)_S \subseteq L(D+Q_1)_S \subseteq \dots \subseteq L(D+Q_1+\dots+Q_h)_S = L(D')_S.$$

Which means, $\dim L(D')_S / L(D)_S =$

$$\sum_{i=1}^{h} \dim L(D+Q_1+\dots+Q_i)_S / L(D+Q_1+\dots+Q_{i-1})_S$$

$$= \sum_{i=1}^{h} d(Q_i) = d(D'_S - D_S)$$

$$= d(D'_S) - d(D_S).$$

- <u>Claim</u>: $\dim_k L(D+Q)_S / L(D)_S = d(Q)$.

Pf: • Let $S = \{P_1, \dots, P_n\}$ & $P_1 = Q$. Let k_Q be the residue field $R_Q/\mathfrak{m}_Q$ with degree $d := d(Q) = [k_Q : k]$.

• Let $x_1', \dots, x_d' \in R_Q$ be a k-basis of k_Q (when seen mod $\mathfrak{m}_Q$).

• By the approx. theorem we can find another

k-basis $x_1, \dots, x_d \in R_Q$ s.t. $\begin{cases} v_Q(x_j - x_j') \geq 1, \ \forall j \\ v_P(x_j) \geq 0, \ \forall j, \ \forall P \in S \setminus \{Q\}. \end{cases}$

• Also, by the approx. theorem we can find an element $u \in K^*$ s.t. $v_P(u) = -\mathrm{ord}_P(D+Q)$, $\forall P \in S$.

• Thus, for any $x \in L(D+Q)_S$ we deduce that $xu^{-1} \in R_Q$ ($\because v_Q(xu^{-1}) = v_Q(x) - v_Q(u) = v_Q(x) + \mathrm{ord}_Q(D+Q) \geq 0$) which means there is a unique expression:

$$xu^{-1} = \sum_{j=1}^{d} a_j x_j + x' \ ; \ a_j \in k \ \& \ x' \in \mathfrak{m}_Q.$$

$$\Rightarrow x = \sum_{j=1}^{d} a_j (x_j u) + (x' u) \ .$$

• Note: $v_Q(x'u) = v_Q(x') + v_Q(u) \geq 1 - \mathrm{ord}_Q(D+Q) = -\mathrm{ord}_Q(D)$. For other $P \in S$, $v_P(x'u) \geq v_P(x) \geq -\mathrm{ord}_P(D)$.

• Thus, $x_i' u \in L(D)_S$.

• Hence, $\dim_k L(D+Q)_S / L(D)_S \leq d$.

• Further, note that: If for nontrivial $a_1, \dots, a_d \in k$ $y := \sum_{j=1}^{d} a_j x_j u \in L(D)_S$, then $v_Q(y) \geq -\mathrm{ord}_Q(D)$. But, $(yu^{-1} \bmod \mathfrak{m}_Q)$ is an invertible element in k_Q, implying that $v_Q(yu^{-1}) = 0$, so $v_Q(y) = -\mathrm{ord}_Q(D+Q)$ which is a contradiction!

• Thus, $\dim_k L(D+Q)_S / L(D)_S \geq d$ as well. $\square$

— This gives us a "weak" estimate for $L(D)$:

Corollary 1: For any $D \leq D'$ in $\mathrm{Div}(C)$,

$$\dim_k L(D')/L(D) \leq d(D') - d(D).$$

Pf: • Let $S := \mathrm{supp}(D') \cup \mathrm{supp}(D)$.

• We have $L(D')/L(D) = L(D')/(L(D') \cap L(D)_S)$ $\cong (L(D') + L(D)_S)/L(D)_S \subseteq L(D')_S / L(D)_S$. $\square$

Corollary 2: $\ell(D) := \dim_k L(D)$ is finite for any $D \in \mathrm{Div}(C)$.

Pf: • Let $D_0 > 0$ be s.t. $D \geq -D_0$. Note that if $x \in L(-D_0)$ then: $v_P(x) \geq -\mathrm{ord}_P(-D_0) \geq 0$ for $\forall P \in C$ & $v_Q(x) > 0$ for some $Q \in C$. Thus, x cannot be in K^*.

- This leaves us with $L(-D_0) = \{0\}$.
- By Cor. 1, $\dim_k L(D)/L(-D_0) \leq d(D) - d(-D_0)$.

$\Rightarrow \dim_k L(D) \leq d(D) + d(D_0)$. $\square$

- We end up with the relation:
$\ell(D') - d(D') \leq \ell(D) - d(D)$ if $D' \geq D$.
Thus, as we pick a "bigger" divisor D', $d(D')$ is expected to "better" estimate $\ell(D')$.

Degree of principal divisors

- For an $x \in K^*$, define the divisor of zeros
$\underline{(x)_0} := \sum \{ v_P(x) \cdot P \mid v_P(x) > 0 \}$, &
the divisor of poles as:
$\underline{(x)_\infty} := -\sum \{ v_P(x) \cdot P \mid v_P(x) < 0 \}$.

▷ $(x) = (x)_0 - (x)_\infty$.