- Our goal now is to study the set of all fns. (in K) whose zeros are at least a given set of pts. in C (smooth proj. curve).

(each pt. $P \in C$ is a max. ideal in S, equivalently, a valuation of K!)

- The formal construct, of much use here, is:

## Divisors of a curve (or K)

- Defn: • The free abelian group $\underline{Div(C)} := \sum_{P \in C} \mathbb{Z} \cdot P$ is the group of divisors of the smooth projective curve C.

(Note: Free group means that the pts. $P \in C$ form the "minimal" generator set of Div(C). Also, each elt. in Div(C) "involves" only finitely many pts. $P \in C$.)

• Any elt. in Div(C) is a divisor of K.

• For any point $P \in C$, the map $\mathrm{ord}_P : Div(C) \to \mathbb{Z}$ is defined via: $D = \sum_{P \in C} \mathrm{ord}_P(D) \cdot P$

▷ $\mathrm{ord}_P$ is a group homomorphism.

- Support of $D$, $supp(D) := \{P \in C \mid ord_P(D) \neq 0\}$.

or, effective

- $D$ is integral or positive if $\forall P \in C$, $ord_P(D) \geq 0$.
- A divisor $D_1$ divides $D_2$ if $D_2 - D_1 \geq 0$. We write this as $D_1 \leq D_2$.
- $D_1, D_2$ are coprime if they have disjoint supports.
- degree of $D$, $d(D) := \sum_{P \in C} ord_P(D) \cdot d(P)$

▷ The map $d : Div(C) \to \mathbb{Z}$, $D \mapsto d(D)$ is a gp. homomorphism.

- Its kernel is called the gp. of divisors of deg 0, denoted by $Div_0(C)$.

- Divisors are interesting because each rational fn. $x \in K^*$ has an associated principal divisor:
$$(x) := \sum_{P \in C} v_P(x) . P$$

▷ $(x)$ is well defined because $v_P(x) \neq 0$ only for finitely many $P \in C$.

- Pf: $\because$ $v(x) = 0$ for all, but finitely many, $v \in \mathcal{C}_K$. $\square$

- Eg. Let $C = Z(x_2^2 x_0 - x_1^3 + x_1 x_0^2) \subset \mathbb{P}^2_k$.

▷ The principal divisor for $f = \frac{x_1}{x_2}$ is:

$$P_1 + P_2 - P_3 - P_4 = (x_1/x_2)$$

where, $P_1 = \langle x_0, x_1 \rangle$ & $P_2 = \langle x_1, x_2 \rangle$ are exactly the "points" in $C$ on which $f$ "vanishes". While $P_3 = \langle x_0 + x_1, x_2 \rangle$ & $P_4 = \langle x_0 - x_1, x_2 \rangle$ are exactly the "points" in $C$ on which $f^{-1}$ "vanishes".

▷ Also, in this example $d((\frac{x_1}{x_2})) = 0$.

We'll <u>later</u> see that it is no coincidence!

- <u>Proposition</u>: The set $\underline{\mathrm{Div}_a(C)} := \{(x) \mid x \in K^*\}$ is a subgroup of $\mathrm{Div}(C)$.

<u>Pf</u>: Simply because $\forall x, y \in K^*$, $(xy) = (x) + (y)$, (which is really a consequence of valuations). $\square$

- It's our long-term goal to understand the properties of $\mathrm{Div}_a(C)$, $\mathrm{Div}_0(C)$ & $\mathrm{Div}(C)$!

- For any $x, y \in K$ & $D \in Div(C)$ we write $\underline{x \equiv y \pmod D}$, if $\underline{(x-y) \geq D}$.

$\triangleright \equiv_D$ is an equivalence relation on $K$ (by valuation axioms).

- We now formalize our motivating question of studying fns. with preassigned zeros:

- <u>Defn</u>: • For any $D \in Div(C)$, define the set of fns.
  $\mathcal{L}(D) := \{x \in k^* \mid (x) \geq -D\}$ and
  $\underline{L(D)} := \mathcal{L}(D) \cup \{0\}$.

  • For any $S \subseteq C$, define a restricted divisor
  $\underline{D_S} := \sum_{P \in S} ord_P(D) \cdot P$. (Sometimes, we need $L(D)_S := \{0\} \cup \{x \in k^* \mid (x)_S \geq -D_S\}$.)

- We have the following immediate properties:

<u>Proposition</u>: (i) $L(D)_S$ is a $k$-vector space.

(ii) $D_1 \geq D_2 \Rightarrow L(D_1)_S \supseteq L(D_2)_S$

(iii) $S \subseteq S' \Rightarrow L(D)_S \supseteq L(D)_{S'}$

(iv) $D_S = D'_S \Rightarrow L(D)_S = L(D')_S$.

- <u>Pf</u>: (i) If $v_P(x), v_P(y) \geq -ord_P(D)$ then $\forall \alpha, \beta \in k$,
  $v_P(\alpha x + \beta y) \geq -ord_P(D)$.

  (ii) If $(x)_S \geq -(D_2)_S$ then $(x)_S \geq -(D_1)_S$. Thus, $x \in L(D_1)_S$.
  $\square$