- Till now the theory was driven by the motivating case of alg. closed k.
- In that case the max. ideals of $A = k[x_1, \dots, x_n]$ were indeed in 1-1 correspondence with points in k^n.

- Now, we want to study any k (esp. $\mathbb{F}_q$). To get the "right" theory we now need to redefine the affine space $\mathbb{A}^n_k$ to be $\text{maxSpec}\, A := \{\mathfrak{m} \mid \mathfrak{m} \trianglelefteq A \text{ is maximal}\}$.

- Observe that the geometric & algebraic parts of the last lectures directly generalize.
- Eg. the Zariski topology on $\mathbb{A}^n_k$ is now given by the base closed sets, for each $f \in A$, $Z(f) := \{\mathfrak{m} \mid \mathfrak{m} \in \text{maxSpec}\, A, f \in \mathfrak{m}\}$.

- Imp. Eg.: A projective curve now is $\text{maxSpec}\, S/I = \{\mathfrak{m} \mid \text{homog. max. ideal } \mathfrak{m} \trianglelefteq S \text{ s.t. } I \subseteq \mathfrak{m}\}$, with homog. prime $I \trianglelefteq S$ & $\dim S/I = 1$.

- The key reason why the algebraic part generalizes is that evaluation of an $f \in A$ at a point $P \in \mathbb{A}^n_k = \max\mathrm{Spec}\, A$ still makes sense via:

$$f : \mathbb{A}^n_k \to \bar{k}$$
$$P \mapsto f \bmod P \in A/P \hookrightarrow \bar{k}$$
$$(\because A/P \text{ is a finite field extn. of } k)$$

- In this way, regular fns., rational fns., morphisms, rational maps, germs, smoothness, all generalize to AVs in $\mathbb{A}^n_k$ (any k).

- This "right" theory makes the case of a smooth projective curve C, with fn. field K, especially nice:

It can be easily seen that the points in C are in 1-1 correspondence with (all) the valuations of K. In fact, we can write $C = \mathcal{C}_K$.
(both geometrically & algebraically)

- One parameter of $P \in C$, that gives us much useful information when k is <u>not</u> alg. closed, is <u>$d(P)$</u> $= [R_P/m_P : k]$

- Eg. Let $C = \mathbb{P}^1_k = \text{maxSpec } S$, for the graded $S = k[x_0, x_1]$. It can be seen that the fn. field of C, $k(C) = k(x_1/x_0) \cong k(x) = K$.

Any pt. $P \in C$, corresponds (1-1) to a valuation v of K. Further, say v corresponds to the uniformizer $f \in k[x]$, which is an irreducible nonconstant polynomial.

Observe that $d(P) = [R_P/m_P : k]$
$= [k[x]_{\langle f \rangle}/\langle f \rangle : k] = [k[x]/\langle f \rangle : k]$
$= \deg(f)$.

$\Rightarrow$ Thus, $d(P)$ "remembers" the number of <u>conjugates of P</u> (which "live" in $\overline{k}$).

- Our goal now is to study the set of all fns. (in K) whose zeros are at least a given set of pts. in C (smooth proj. curve).

(each pt. $P \in C$ is a max. ideal in S, equivalently, a valuation of K!)

- The formal construct, of much use here, is:

Divisors of a curve (or K)

- Defn: • The free abelian group $\underline{Div(C)} := \sum_{P \in C} \mathbb{Z} \cdot P$ is the group of divisors of the smooth projective curve C.

(Note: Free group means that the pts. $P \in C$ form the "minimal" generator set of Div(C). Also, each elt. in Div(C) "involves" only finitely many pts. $P \in C$.)

• Any elt. in Div(C) is a divisor of K.

• For any point $P \in C$, the map $ord_P : Div(C) \to \mathbb{Z}$ is defined via: $D = \sum_{P \in C} ord_P(D) \cdot P$

▷ ord_P is a group homomorphism.