- Note that we want to construct a fn., with preassigned zeros, over a <u>projective</u> curve.

- Eg. The curve $Z(x_2^2 - x_1^3 + x_1)$ needs to be first <u>projectivized</u> to
$$C := Z(x_2^2 x_0 - x_1^3 + x_1 x_0^2) \subseteq \mathbb{P}_k^2 .$$
Now consider the pt. $P = (1,0,0)$, that corresponds to $(0,0)$ of the affine curve.
$\triangleright\ \nexists\ f \in K$ s.t. zeros of f are exactly P!

The issue can be seen by taking $f = x_1/x_2$.
Its valuation at P is $v_P(x_1) - v_P(x_2) = 2 - 1 = 1$.
But, f also has a root $Q = (1,1,0)$ of order
$$v_Q(x_1) - v_Q(x_2) = 0 - 1 = -1 .$$
[$\because$ x_1^{-1} is defined at the pt. Q.]

- Can we find a rational fn. with a preassigned set of zeros/poles (i.e. valuations)?

[Approximation Thm.]

- **Theorem**: Let K be the fn. field of a curve C.

Chinese Remaindering on valuations?

Let $p_1, \ldots, p_h \in C$ be distinct, with corresponding valuations $v_1, \ldots, v_h \in G_K$. Let $u_1, \ldots, u_h \in K$ and $m_1, \ldots, m_h \in \mathbb{Z}$.

Then, $\exists u \in K$ s.t. $\forall i$, $v_i(u - u_i) \geq m_i$.

- Pf: • Claim: $v_1, \ldots, v_h$ are $\mathbb{Q}$-independent, i.e. $\nexists$ nonzero $(r_1, \ldots, r_h) \in \mathbb{Q}^h$ s.t. $\forall z \in K$, $\sum_{i=1}^{h} r_i v_i(z) = 0$.

Pf: (Exercise)

- We want to construct u as $\sum_{i=1}^{h} x_i u_i$, in steps.
- Using the above fact we first construct $z_i \in k^*$ s.t. $\det((v_i(z_j))) \neq 0$.
- Start with $z_1 \in K^*$ s.t. $v_1(z_1) \neq 0$. Thus, $\mathrm{rk}_{\mathbb{Q}} \{(r_1, \ldots, r_h) \in \mathbb{Q}^h \mid \sum r_i \cdot v_i(z_1) = 0\} = (h-1)$.
- Pick a nonzero $(r_1, \ldots, r_h)$ from the above space.

(claim) $\Rightarrow \exists z_2 \in K^*, \sum_{i=1}^{h} r_i v_i(z_2) \neq 0$

- Next, consider the space $\{(r_1, \dots, r_h) \in \mathbb{Q}^h \mid \forall j \in [2], \sum_{i=1}^{h} r_i v_i(z_j) = 0\}$ with $rk_{\mathbb{Q}} \leq (h-2)$ & continue.

- This process gives us $z_1, \dots, z_h \in K^*$ s.t. no nonzero $\mathbb{Q}$-li. combination of $\{v_i(z_j)\}_i$ vanishes simultaneously for all $j \in [h]$.

$\Rightarrow \det((v_i(z_j))) \neq 0$.

- Solve for c's s.t. $\sum_{j=1}^{h} c_{jm} v_i(z_j) = \begin{cases} -1, & \text{if } i=m \\ 1, & \text{else} \end{cases}$

- Pick a $d \in \mathbb{N}$ s.t. $\{dc_{jm} \mid 1 \leq j, m \leq h\}$ are integral.
- Define $y_m := \prod_{1 \leq j \leq h} z_j^{dc_{jm}}$.

$\Rightarrow v_i(y_m) = \begin{cases} -d, & \text{if } i=m \\ d; & \text{else} \end{cases}$.

- Define $x_m := (1 + y_m^{-1})^{-1}$.

$\Rightarrow$ For $i \neq m$, $v_i(x_m) = d$. For $i = m$, $v_i(x_m - 1) = d$.

- Fix d so large s.t. $d+v_i(u_j) \geq \max\{m_1, \ldots, m_h\}$ for all $1 \leq i,j \leq h$.
- Finally, set $u := \sum_{i=1}^{h} x_i u_i$,

$$\Rightarrow u - u_i = u_i(x_i - 1) + \sum_{\substack{1 \leq m \leq h \\ m \neq i}} x_m u_m$$

- $\because v_i(u_i(x_i - 1)) = d + v_i(u_i) \geq m_i$

& for $i \neq m$, $v_i(u_m x_m) = d + v_i(u_m) \geq m_i$.

$\Rightarrow v_i(u - u_i) \geq m_i$. $\square$

- We also deduce, for later use,

- <u>Corollary</u>: Let S be a finite set of pts. in a smooth curve C with fn. field K. Let $\{m_P \mid P \in S\} \subseteq \mathbb{Z}$. Then, $\exists f \in K$ s.t. $\forall P \in S$, $v_P(f) = m_P$.

<u>Pf</u>:
- $\forall P \in S$, pick a $u_P \in K^*$ st. $v_P(u_P) = m_P$.
- Using the Theorem, construct $u \in K^*$ s.t. $v_P(u - u_P) > m_P$.
- Observe that $v_P(u) = v_P((u - u_P) + u_P)$ which equals $v_P(u_P) = m_P$ because $v_P(u - u_P) > v_P(u_P)$.
- Thus, u satisfies the conditions on f. $\square$