- Now we study only non-singular curves.
- The following terms, usually, mean the same:

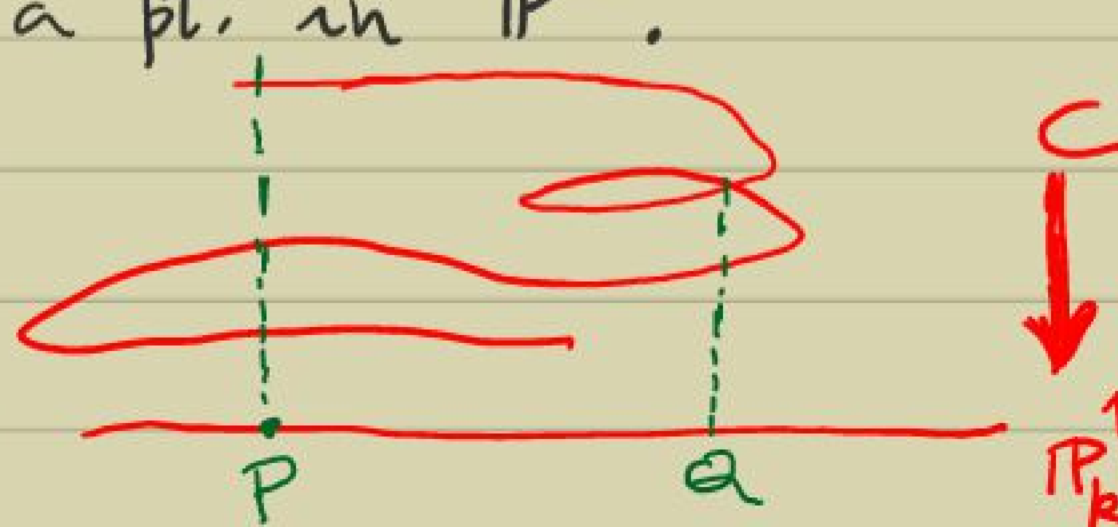
non-singular ≈ simple ≈ <u>smooth</u>

Points on a smooth curve

- Let C be a smooth curve with fn. field K/k.
- If $x \in K \setminus k$, then $k(x)$ can be viewed as the fn. field of the projective line $\mathbb{P}^1$.
- We now know:

▷ Any pt. in $\mathbb{P}_1$ corresponds to a dvr in $k(x)$, which <u>extends</u> to one in K, hence a pt. in C.

▷ Any pt. in C corresponds to a dvr $R \subset K$, so the <u>restriction</u> $R \cap k(x)$ is a dvr in $k(x)$, hence a pt. in $\mathbb{P}^1$.



<u>Covering</u> of the projective line

(alg. closed k)

- Let C be a nonsingular curve with fn. field K. We know $C \cong C_K$. So we can define natural valuations wrt each $p \in C$.

Note that v_p's are distinct.

Defn: • A pt. $p \in C$ defines the valuation v_p on K given by the valuation ring $\mathcal{O}_{C,p}$.

• The field $\mathcal{O}_{C,p}/m_p =: k_p$ is called the residue field at p.

• The degree of p is $d(p) := [k_p : k]$.

- Proposition: (i) $d(p) < \infty$.

(ii) $\bigcap_{i \geq 0} m_p^i = \langle 0 \rangle$.

(iii) If $m_p = \langle u \rangle . \mathcal{O}_{C,p}$ then $\forall \alpha \in K$, $v_p(\alpha) =$ the largest $i \in \mathbb{Z}$ s.t. $\alpha \in u^i . \mathcal{O}_{C,p}$.

- Pf: • (i) follows from the fact that if u is the uniformizer (i.e. generating m_p) then, $k(u) \subset K$ is a finite algebraic extn.

• (ii) $\forall x \in \mathcal{O}_{C,p}$, only finitely many $u^i | x$.

• (iii) Clear, by the way we defined v_p using u. $\square$

- Can we find a rational fn. with a preassigned set of <u>zeros</u>/<u>poles</u> (i.e. valuations)?

[<u>Approximation Thm.</u>]

- <u>Theorem</u>: Let K be the fn. field of a curve C.

Chinese remaindering on valuations?

Let $p_1, \ldots, p_h \in C$ be distinct, with corresponding valuations $v_1, \ldots, v_h \in C_K$. Let $u_1, \ldots, u_h \in K$ and $m_1, \ldots, m_h \in \mathbb{Z}$.

Then, $\exists u \in K$ s.t. $\forall i$, $v_i(u - u_i) \geq m_i$.

- <u>Pf</u>: • <u>Claim</u>: $v_1, \ldots, v_h$ are $\mathbb{Q}$-independent, i.e. $\nexists$ nonzero $(r_1, \ldots, r_h) \in \mathbb{Q}^h$ s.t. $\forall z \in K$, $\sum_{i=1}^{h} r_i v_i(z) = 0$.

Pf: (<u>Exercise</u>)

• We want to construct u as $\sum_{i=1}^{h} x_i u_i$, in steps.