Existence of non-singular models

<u>Theorem</u>: Let $k \subset K$ be a trdeg one field. Then, the abstract curve C_K is isomorphic to a non-singular projective curve.

- <u>Pf</u>: • <u>Idea</u>: Glue the various dvr's, using the Segre embedding, into a projective curve.

• $\boxed{\text{Claim}}$: If $R \subset K$ is a dvr with max. ideal $\mathfrak{m}$, then R can be seen as germs of a non-sing. AV.

• <u>Pf</u>: - Let $\mathfrak{m} = \langle y \rangle_R$ & B be the integral closure of $k[y]$ in K. Let $\mathfrak{n} := B \cap \mathfrak{m}$.

- Deduce that B is an integrally closed domain in R & $\mathfrak{n}$ is a max. ideal of B.

$\Rightarrow$ $B_{\mathfrak{n}}$ is a dvr isomorphic to R.

$\Rightarrow$ R can be seen as the local ring of the non-sing. AV (corr. to B) at the point (corr. to $\mathfrak{n}$). $\square$

($\because B$ is integrally closed)

• Thus, for a $v \in C_K$, there is a <u>non-singular</u> affine curve V & a point $q \in V$ s.t. $R_v \cong \mathcal{O}_{V,q}$. Clearly, $k(v) = k$.

- $\boxed{\text{Claim}}$: V is isomorphic to an open subset of C_K. (I.e. germs on V are all, but finitely many, elements of C_K.)
- Pf: A dvr R_v ($v \in C_K$) appears as a germ on V iff $B \subseteq R_v$

  iff $y \in R_v$.

How many R_v's can y avoid?

If $y \notin R_v$ then $y^{-1} \in \mathfrak{m}_v$, so y can only avoid <u>finitely many</u> v's. $\square$

- Thus, each $v \in C_K$ has an open nbd. isomorphic to a non-singular affine curve.
- Cover C_K by a <u>finite</u> number of such open nbds. $\{U_i\}$. The above gives us a "good" morphism $\varphi_i : U_i \to Y_i$ for a non-singular projective curve Y_i.
- Extend this to a <u>birational</u> map φ from C_K <u>into</u> $\prod Y_i$ (<u>Segre</u> product)
- $Y := \varphi(C_K)$ is our non-sing. projective curve. $\square$