- For two distinct (monic) irreducibles f, f' one can see $v_f \neq v_{f'}$.
  e.g. $f^{-1} \notin R_f$ but $f^{-1} \in R_{f'}$.
- <u>No other valuations?</u>
  Let R be any valuation ring in K, with maximal ideal $\mathfrak{M}$ & valuation v.

  <u>Case I:</u> $x \in R$.

  $\Rightarrow k[x] \subseteq R$

  $\Rightarrow k[x] \cap \mathfrak{M} \neq 0$ (else, $k[x]^* \subseteq R \setminus \mathfrak{M}$, $\Rightarrow v = 0$!)

  $\Rightarrow k[x] \cap \mathfrak{M}$ is a nonzero prime ideal of $k[x]$.

$\Rightarrow \exists$ irreducible f s.t. $k[x] \cap \mathfrak{M} = \langle f \rangle_{k[x]}$

  $\Rightarrow \mathfrak{M} = \langle f \rangle_R$ (Why?)

  $\Rightarrow v = v_f$

  <u>Case II:</u> $x \notin R$.

  $\Rightarrow k[x^{-1}] \subseteq R$

  $\Rightarrow$ (as before) $R_{x^{-1}} = R$. $\square$

Extension of valuation rings

- Any field K of $\mathrm{trdeg}_k K = 1$ can be written as $\underbrace{k \subset k(x)}_{x \text{ transcendental over } k} \underbrace{\subseteq K}_{\text{finite extension}}$.

- We will show how valuations of K are related to those of $k(x)$.

- **Theorem:** Let $\mathrm{trdeg}_k K = 1$; $k \subseteq R$ be a subring of K & m_R be a nontrivial ideal of R. Then, $\exists$ valuation ring B of K with maximal ideal m_B s.t. $m_R \subseteq m_B \cap R$.

- **Pf sketch:** • The proof is nonconstructive.
  - For any subring $R \subseteq R' \subseteq K$ define an ideal of R', $m_R \cdot R' := \langle m \cdot r \mid m \in m \ \& \ r \in R' \rangle$.
  - Consider the subring family: $\mathcal{F} := \{R' \text{ subring of } K \mid R \subseteq R' \ \& \ m_R \cdot R' \neq R'\}$.
  - Clearly, $R \in \mathcal{F}$.

- Show that $\mathcal{F}$ has maximal elements (wrt the subring ordering).
- Show that any maximal element satisfies the required conditions on B. $\square$

— Eg. In the case of $X = Z(x_2^2 - x_1^3)$, the ring $R := A(X)_{\langle x_1, x_2\rangle}$ is not a dvr in $K(X)$. But it can be seen that the above process points us to the subring $B = k[\frac{x_2}{x_1}, x_1]_{\langle \frac{x_2}{x_1}, x_1\rangle}$ of $K(X)$ which is a dvr.

- Moreover, $\langle \frac{x_2}{x_1}, x_1\rangle \cdot B \cap R = \langle x_1, x_2\rangle \cdot R$.

— This process is, thus, guaranteed to make an affine curve X birational to a nonsingular curve, when X has only one non-simple pt.

— What do we do when X has multiple non-simple points?

- We GLUE several dvr's to form a curve!