- Proposition 1: If $\underset{\text{max. ideal}}{\mathfrak{m}}$ ... $k \subset \mathfrak{m} \subset R \subset K$, then there is a valuation $v: K^* \to \mathbb{Z}$. Further,

max. ideal, dvr, frac-field

(valuation ring of v) ($v(0) := \infty$)

$$R = \{x \in k \mid v(x) \geq 0\}, \text{ and}$$
$$\mathfrak{m} = \{x \in k \mid v(x) > 0\}.$$

Pf. Sketch:
- Let $\mathfrak{m} = \langle u \rangle$, where u is called the uniformizer.
- Express any $x \in k$ as $x = u^e \cdot x'$, $x' \in R \setminus \mathfrak{m}$.
- Define $v(x) := e$. $\square$

- Exercise: If a field K/k has a valuation, then $\exists$ corresponding local domain R of $\dim_k = 1$.

- Property III: The dvr $R \subset K$ has the property that
- Defn: it is "integrally" closed:

If any polynomial $f(X) := X^n + a_1 X^{n-1} + \cdots + a_n \in R[X]$ has a root $\alpha \in K$, then $\alpha \in R$.

(I.e. $\alpha \in K$ with monic minimal polynomial over R is in R)

Eg. $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$.

- Thus, to resolve the singularity we just need to pick the integral closure of $\mathcal{O}_{X,P} = (k[x_1,x_2]/\langle x_2^2 - x_1^3\rangle)_{\langle x_1,x_2\rangle}$ in $k(X) = k(x_1)[x_2]/\langle f\rangle$.

- <u>Proposition 2</u>: For a local domain R with fraction field K of $\mathrm{trdeg}_k K = 1$, TFAE:

  (i) R is a dvr.

  (ii) K has a valuation with valuation ring R.

  (iii) R is integrally closed in K.

- <u>Pf Sketch</u>: (ii) $\Rightarrow$ (iii): Let $v : K^* \to \mathbb{Z}$ be the valuation.
  - Say, an $x \in K$ satisfies the equation
    $x^n + a_1 x^{n-1} + \dots + a_n = 0$ with a_i's in R.

    $\Rightarrow v(x^n) = v(a_1 x^{n-1} + \dots + a_n) \geq v(a_i\, x^{n-i}) \geq v(x^{n-i})$,
    where $a_i x^{n-i}$ has the min. $v(\cdot)$.

    $\Rightarrow n \cdot v(x) \geq (n-i)\, v(x) \quad \Rightarrow i \cdot v(x) \geq 0$

    $\Rightarrow v(x) \geq 0$. Thus, $x \in R$.
  - Moreover, the least value element in $\mathfrak{M} := \{x \in K \mid v(x) > 0\}$ generates the unique maximal ideal $\mathfrak{M}$.

(iii) $\Rightarrow$ (i): We will show a "linear term" in the relationship between any $x_1, x_2 \in \mathfrak{M}$.

- Say, $x_1^{e_1}\cdot(\alpha_1+f_1(x_1,x_2)) = x_2^{e_2}\cdot(\alpha_2+f_2(x_1,x_2))$ where $e_1 \le e_2 \in \mathbb{N}$; $\alpha_1,\alpha_2\in k^*$ & $f_1,f_2\in\langle x_1,x_2\rangle$.
- If $e_1=0$ then $e_2=0$ (else, x_2 becomes a unit in R, violating $\mathfrak{M}\subsetneq R$).
- So, $0<e_1\le e_2$.

$\Rightarrow$ $x_1/x_2 \in K$ is a root of

$$Y^{e_1} = x_2^{e_2-e_1}\cdot\left(\frac{\alpha_2+f_2}{\alpha_1+f_1}\right)$$

which is a monic poly. in $R[Y]$.

$\Rightarrow$ $x_1/x_2\in R$ ($\because$ R is integ. closed).

$\Rightarrow$ $\langle x_1,x_2\rangle\cdot R = \langle x_2\rangle\cdot R$.

- By repeating this process we can deduce that $\mathfrak{M}$ has a single generator.

$\Rightarrow$ R is a dvr. $\square$

We will now follow the plan: Valuations of $K(\mathbb{A}^1)$ $\rightsquigarrow$ valuations of $K(X)$ $\rightsquigarrow$ "glue" the dvr's to define a nonsingular projective $\tilde{X}$ birational to X!

Valuations of $K = k(x)$.

- Before understanding valuations of all trdeg $=1$ fields, we characterize those of the pure transcendental extn. $k(x)/k$.

Defn: For an irreducible $f \in k[x]$, define the subring (of K), $\underline{\underline{R_f}} := \{g/h \in K \mid f \nmid h\}$.

- Theorem: The (distinct) valuation rings of K are: R_f, for irreducible $f \in k[x]$ & $R_{x^{-1}}$, (viewing x^{-1} as an irreducible polynomial in $k[x^{-1}]$).

Pf: • These rings define valuation on K^*: eg. for irred. $f \in k[x]$, some $\alpha \in K = k(x)$, define $v_f(\alpha) := e$, where $\alpha = f^e \cdot \frac{\beta}{\gamma}$ for some $\beta, \gamma \in k[x]$ coprime to f.

• For $f = x^{-1}$, express $\alpha = \left(\frac{1}{x}\right)^e \cdot \frac{\beta(x^{-1})}{\gamma(x^{-1})}$ for $\beta, \gamma \in k[y]$ with $\beta(0) \cdot \gamma(0) \neq 0$.