Resolving singularity (Blowing-up!)

- To optimally exploit the tangent spaces, we would like to show that every curve is birational to a non-singular projective curve!

- How do we resolve the $P=\bar{0}$ singularity of $X = Z(\underbrace{x_2^2 - x_1^3}_{=:f})$?

- We have $k \subset k[x_1,x_2]/\langle f \rangle \subseteq A(X)_{m_p} \subset k(X)$.

- If we can find a local domain "above" $A(X)_{m_p}$ that looks like $A(\tilde{X})_{m_p}$ for some AV $\tilde{X}$ s.t. $k(\tilde{X}) = k(X)$, then we can study $\tilde{X}$ instead of X for "local" purposes.

- Let $y := x_2/x_1 \in K(X)$ and consider $A(\tilde{X}) := k[x_1, y]/\langle \underbrace{y^2 - x_1}_{=:\tilde{f}} \rangle$. Clearly, the corr. AV $\tilde{X}$ has $(0,0)$ simple!

- We will now spend much time in understanding the relationship between $\tilde{X}$ & the original X, in general terms. (Note: $A(X)_{\langle x_1,x_2\rangle} \subset A(\tilde{X})_{\langle x_1,y\rangle}$ & $k(X)=k(\tilde{X})$.)

- The process is called <u>blowing up</u> because a point (x_1,x_2) is being "mapped" to: (x_1, yx_1, y). [eg. $(0,0)\mapsto(0,0,1)$]

- <u>Property I</u>: $A(\tilde{X})_{\langle x_1,y\rangle}$ is a local domain with the maximal ideal <u>principal</u>, while $A(X)_{\langle x_1,x_2\rangle}$ was not.

  The maximal ideal of the former is $\langle x_1,y\rangle = \langle y^2,y\rangle = \langle y\rangle$, while that of the latter is $\langle x_1,x_2\rangle$ which can be seen to have no single generator.

- <u>Defn</u>: A ring R is a <u>dvr</u> (discrete valuation ring) if it is local, domain & max. ideal is principal.

- Property II: Say, we are in a situation $k \subset \mathfrak{m} \subset R \subset K$, where R is a dvr with max. ideal $\mathfrak{m}$, and fraction field K. Then, there is a way to define a degree-like map $v : K^* \to \mathbb{Z}$.

- Defn: A discrete valuation of K is a map $v : K^* \to \mathbb{Z}$ s.t. for all $x, y \in K^*$,

$$v(xy) = v(x) + v(y),$$
$$v(x+y) \geq \min(v(x), v(y)).$$

[Note the diff. from $\deg(\cdot)$]

- In the previous eg. $k \subset \underset{\mathfrak{m}}{\overset{\shortparallel}{\langle x_1, y\rangle}} \triangleleft \underset{R}{\overset{\shortparallel}{(k[x_1,y]/\langle y^2 - x_1\rangle)}}_{\langle x_1, y\rangle}$

$$\subset K := k(x_1)[x_2]/\langle x_2^2 - x_1^3\rangle.$$

- any element α in K looks like $\dfrac{a(x_1) + x_2 \cdot b(x_1)}{c(x_1)}$ for $a, b, c \in k[x_1]$.
- Define $v(\cdot)$ using the y (called uniformizer) as: Express $\alpha = y^e \cdot \alpha'$ where, $e \in \mathbb{Z}$ & $\alpha' \in R \setminus \mathfrak{m}$. Then, $v(\alpha) := e$.
- Eg. $v(x_1) = v(y^2) = 2$; $v(x_2) = v(yx_1) = 3$;
$v(1 + x_1) = 0$; $v(1/x_1) = -v(x_1) = -2$.