

$$\forall i \in [t], \quad \sum_{j=1}^n (\partial_j f_i)|_P \cdot x_j = 0.$$

- Clearly, its solution space is of  $\text{rk} = n - \text{rk}((\partial_j f_i)|_P)$ .  $\square$

- Now we show that the dual tangent space is related to the germs (at  $P = \bar{0}$ ).

- Proposition 2:  $A^{(1)}/\mathcal{I}(X)^{(1)} \cong M_P/M_P^2$ .

Pf: • Consider the natural map  $\phi: A^{(1)}/\mathcal{I}(X)^{(1)} \rightarrow M/M^2$ ;  $f \mapsto f$ . (Note  $f(\bar{0}) = 0$ .)

- Clearly,  $\phi$  is a ( $k$ -vector-space) homomorphism.

- injectivity:  $\phi(f) = 0 \Rightarrow$  the linear form  $f \in M^2 \Rightarrow f = 0$  in  $A(X) \Rightarrow f = 0$  in  $A^{(1)}/\mathcal{I}(X)^{(1)}$ .

- surjectivity: Let  $\tau = f/g \in M_P$ . Wlog  $g(P) = \underline{1}$ .

Set  $\tau' := \sum_{i=1}^n (\partial_i \tau)|_P \cdot x_i$ . We will show that  $\phi(\tau') = \tau$ .

$$\bullet g(\tau - \tau') = f - g \cdot \sum_{i=1}^n \left( \frac{\partial_i f \cdot g - f \cdot \partial_i g}{g^2} \right) \Big|_P \cdot x_i$$

$$= f - g \cdot \sum_{i=1}^n (\partial_i f)|_P \cdot x_i \equiv_{(\text{mod } M_P^2)} f - \sum_{i=1}^n (g \cdot \partial_i f)|_P \cdot x_i$$

• The last step uses that both  $g - g(P)$  &  $x_i$  are in  $M_P$ .

• So, we arrive at  $f - \sum_{i=1}^n (\partial_i f)|_P \cdot x_i$  which is a polynomial with no linear part.

• Thus, it is  $0 \pmod{M_P^2}$ .

$$\Rightarrow g \cdot (\tau - \tau') \equiv 0 \pmod{M_P^2}.$$

$$\Rightarrow \tau - \tau' \equiv 0 \pmod{M_P^2}.$$

$$\Rightarrow \varphi(\tau') = \tau, \text{ proving surjectivity. } \square$$

— Summing up, the tangent space has the following relations:

$$\begin{aligned} \text{— Corollary: } \operatorname{rk} T_{X,P} &= n - \operatorname{rk} ((\partial_i f_j)|_P) = n - \operatorname{rk} A^{(1)}/I(X)^{(1)} \\ &= n - \operatorname{rk} M_P/M_P^2. \end{aligned}$$

— Further, since  $T_{X,P}$  "forgets" some of the conditions that define  $X$ , we have:

$$\text{— Proposition 3: } \operatorname{rk} T_{X,P} \geq \dim X.$$

Pf: (Exercise) □

← We want to give special attention to equality.....

- Definition:  $X$  is non-singular at a point  $P$   
 if  $\text{rk } T_{X,P} = \dim X$ . ( $P$  is simple.)  
 $X$  is non-singular if there are no singularities.

- eg.  $X = Z(x_2 - x_1^3) \subseteq \mathbb{A}^2$  is non-singular at  $P = \bar{0}$ ;  
 $\text{rk } (\partial_j f)|_P = \text{rk } [-3x_1^2 \ 1]|_P = 1 = n - \dim X$ .

- eg.  $X = Z(x_2^2 - x_1^3) \subseteq \mathbb{A}^2$  is singular at  $P = \bar{0}$ ;  
 $\text{rk } (\partial_j f)|_P = \text{rk } [-3x_1^2 \ 2x_2]|_P = 0 < n - \dim X$ .

But non-singular at  $P = (1, 1)$ :

$\text{rk } (\partial_j f)|_P = \text{rk } [3 \ 2] = 1 = n - \dim X$ .

true for  
 any  $k$  and  
 any nonzero  
 $P \in X$

- A real picture:

