- Eg. • The AV $X = Z(x_2 - x_1^3) \subseteq \mathbb{A}_k^2$ is isomorphic to $\mathbb{A}_k^1$.

• The AV $X = Z(x_2^2 - x_1^3) \subseteq \mathbb{A}_k^2$ is <u>not</u> isomorphic to $\mathbb{A}_k^1$.

- Surprisingly,

<u>Proposition 1</u>: Any AV of dimension r is isomorphic to a hypersurface in $\mathbb{A}^{r+1}$.

<u>Pf</u>:

• Consider an AV X for a prime $I(X) \trianglelefteq A$.

• Say, $r = \dim A/I(X) = \operatorname{trdeg}_k k(A(X))$.

• Wlog, assume $x_1, \dots, x_r \in k(A(X))$ are algebraically independent over k.

$$\Rightarrow \underbrace{k \subset k(x_1, \dots, x_r)}_{\text{pure transcendental}} =: \underbrace{k' \subseteq k(A(X))}_{\text{finite}}.$$

Primitive Element Theorem

• It can be shown that the finite extension has to be of the form $k' \subseteq k'(\alpha) = k'[y]/\langle f(y) \rangle$ for some $f \in k'[y]$.

- Thus, $k(A(X)) \cong k(x_1,..,x_r)[y]/\langle f(y)\rangle$.
- (By Proposition 2) $\Rightarrow$
  $X \cong Y := Z(f) \subseteq \mathbb{A}_k^{r+1}$. $\square$

- Let $k = \overline{\mathbb{F}_p}$. The <u>Primitive Element</u> theorem states that "any" finite extn. $k \subseteq L$ (above k) can be expressed as:
  $L \cong k[y]/\langle f(y)\rangle$.
- <u>Eg.</u> Say, we have $k \subseteq \underbrace{k \subseteq k(\alpha_1,\alpha_2)}_{L}$.

- Try out various $c \in k$ & check whether
  $k(\alpha_1 + c\alpha_2) \stackrel{?}{=} k(\alpha_1,\alpha_2)$.
- It is easy to see that all but finitely many c work!
- (The proof requires us to choose k carefully.)

Non-singular varieties

- Let $X \subseteq \mathbb{A}^n$ be an AV of dim $= r$.
- Let $I(X) = \langle f_1, \dots, f_t \rangle \trianglelefteq A$ and $P \in X$.

- We want to study X at P via the tangents. To reduce the non-linear structure of X to a linear one, we view the tangent space as a vector space.
- Wlog assume $P = \bar{0} \in X$.

- Defn: • The tangent space gives a "local approximation" of X at P and can be defined as $T_{X,P} := Z(I(X)^{(1)})$, where $f^{(1)}$ denotes the deg $= 1$ part of f & $I(X)^{(1)} := \langle f_i^{(1)} \mid i \rangle$.
  • The linear functions on the tangent space give us the dual space $A^{(1)}/I(X)^{(1)}$.

- From the duality we can deduce:

- Proposition 1: $\operatorname{rk} T_{X,P} + \operatorname{rk} ((\partial_j f_i))|_P = n$.

Pf: • The linear space $T_{X,P}$ is given by the system of equations: