- Varieties X, Y are <u>isomorphic</u> (<u>birationally</u>) if $\exists$ rational map $\Phi: X \to Y$ which has an inverse rational map $Y \to X$.

<u>Eg.</u>: • Let $Y \subseteq \mathbb{A}^n$ be a hypersurface $\{p \mid f(p)=0\}$.
• Define $H := \{\bar{x} \in \mathbb{A}^{n+1} \mid x_{n+1} \cdot f(x_1, \dots, x_n) = 1\}$.
• Then,

$$\begin{aligned} \Phi: \mathbb{A}^n \setminus Y &\to H \quad \text{is an isomorphism.} \\ p &\mapsto (p, f(p)^{-1}) \end{aligned}$$

- <u>Defn</u>: Above, $\underset{X_f \,=}{\mathbb{A}^n \setminus Y}$ "is" an affine variety. Such objects we will call <u>affine open</u>.

• Its <u>coordinate ring</u> is

$$R[\bar{x}] / \langle x_{n+1} \cdot f(x_1, \dots, x_n) - 1 \rangle \cong \overset{\mathcal{O}_X(X_f) \,=}{A_f}.$$

- <u>Prop. 1</u>: Any variety X has a base consisting of affine opens.

<u>Pf</u>: Since, $\{X_f \mid f \in A(X)\}$ form a base of the open subsets. $\square$

- Birational isomorphism has a nice algebraic criterion:

- <u>Proposition 2</u>: Varieties $X \cong Y$ iff $k(X) \cong k(Y)$.

<u>Pf</u>:

- Let $\phi: X \to Y$ be the isomorphism.
- Pick a dominant rational map (U, ϕ_U).
- Let $(V, f) \in k(Y)$ be a rational fn.
- $\phi_U^{-1}(V)$ is open & non-empty in X.
- Thus, $(\phi_U^{-1}(V), f \circ \phi_U)$ is a rational fn. in $k(X)$.

$\Rightarrow$ We get an isomorphism from $k(Y)$ to $k(X)$.

Conversely, let $\psi: k(Y) \to k(X)$ be a k-algebra isomorphism.

- In particular, ψ identifies the maximal ideals of $A(Y)$ (resp. $S(Y)$) with those of $A(X)$ (resp. $S(X)$).

$\Rightarrow X, Y$ are isomorphic. $\square$