- Let us redefine the ring of regular fns. on a variety X as $\mathcal{O}_X(X)$ & its local version at a point $P \in X$ as $\mathcal{O}_{X,P}$ (i.e. germs).

- <u>Claim</u>: If $f \in \mathcal{O}_X(X)$ then there are $g, h \in A(X)$ s.t. $f = g/h$ on X.

<u>Pf</u>:

- Say, $f = g_i/h_i$ on open $U_i \subseteq X$, for $i \in \{1,2\}$ and $g_i, h_i \in A(X)$.

- As U_1, U_2 are nonempty, $U_1 \cap U_2$ is a nonempty open subset of X.

- Thus, $f = g_1/h_1 = g_2/h_2$ on $U_1 \cap U_2$.
  $\Rightarrow g_1h_2 - g_2h_1 = 0$ on $U_1 \cap U_2$.

- As k is alg. closed, deduce $g_1h_2 - g_2h_1 = 0$ on U_1 & U_2. Thus, $f = g_1/h_1$ on X. $\square$

Lecture 7

- For an open $U \subseteq X$ of an affine variety we denote by $\mathcal{O}_X(U)$ the ring of regular functions on U.

- Let us compute this for a special U: For an $f \in A(X)$, $X_f := \{P \in X \mid f(P) \neq 0\}$.

$\{X_f \mid f\}$ form a base.

- <u>Defn</u>: X_f is called a <u>distinguished open subset</u>. Any open $U \subseteq X$ can be written as a union of X_f's.

- We will relate $\mathcal{O}_X(X_f)$ with $A(X)_f := \{ g/f^r \mid g \in A(X), r \in \mathbb{N}\} = (f)^! \cdot A(X)$

<u>Proposition 1</u>: $\mathcal{O}_X(X_f) = A(X)_f$.

<u>Pf</u>:

- $A(X)_f \subseteq \mathcal{O}_X(X_f)$ is natural.

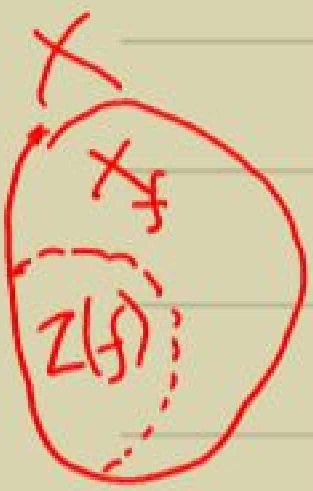
- For the converse, let $\varphi \in \mathcal{O}_X(X_f) \subseteq k(X)$. Let $J := \{g \in A(X) \mid g \cdot \varphi \in A(X)\}$.

φ has the same representation on each open set.

- We want to show $\exists r$, $f^r \in J$.

- Note that J is an ideal of $A(X)$.

- For any $P \in X_f$, since φ is a rational fn. in its nbd., J contains a g s.t. $g(P) \neq 0$



- Thus, $Z(I(X)+J)$ cannot have $P \in X_f$.

$$\Rightarrow \quad Z(I(X)+J) \subseteq Z(f).$$

$$\Rightarrow \quad I(Z(f)) \subseteq I(Z(I(X)+J)).$$

- By Hilbert Nullstellensatz, $\exists r,\ f^r \in I(X)+J$. $\square$

<u>Corollary:</u> $\mathcal{O}_X(X) = A(X)$.

<u>Pf:</u> $\mathcal{O}_X(X) = \mathcal{O}_X(X_1) = A(X)_1 = A(X)$. $\square$

- Now we want to <u>localize</u> this property to a <u>single</u> point $P \in X$.

- By abuse of notation, for $P \in X$,
$\mathfrak{m}_P := \{ f \in A(X) \mid f(P) = 0 \}$.

▷ $\mathfrak{m}_P \trianglelefteq A(X)$ is maximal.

- Thus, we can localize: $A(X)_{\mathfrak{m}_P}$, &:

- Proposition 2: $\mathcal{O}_{X,P} = A(X)_{\mathfrak{m}_P}$.

Pf:

- Clearly, $A(X)_{\mathfrak{m}_P} \subseteq \mathcal{O}_{X,P}$.

- Let $(U, f) \in \mathcal{O}_{X,P}$.

$\Rightarrow$ $f = g/h$ on U for some $g, h \in A(X)$.

- Clearly, $h(P) \neq 0$.

$\Rightarrow h \in A(X) \setminus \mathfrak{m}_P$.

- Thus, $f = g/h \in A(X)_{\mathfrak{m}_P}$. $\square$