- We will now see the rings $\mathcal{O}(Y)$ & $\mathcal{O}_p(Y)$ in more <u>explicit</u> terms.

- <u>Proposition 1:</u> For an AV Y, & alg. closed k, the natural map $\phi: A(Y) \to \mathcal{O}(Y)$ is an isomorphism.

- <u>Pf:</u> • An $f \in A(Y)$ defines a fn. $f: Y \to k$. Thus, $f \in \mathcal{O}(Y)$ makes sense.

• Conversely, if $f \in \mathcal{O}(Y)$ then it can be seen (refer next lecture) that $f = g/h$ for some $g, h \in A$; i.e. f has a "global" rational representation.

• As f is defined on Y, we deduce that h has <u>no</u> zero in Y.

• Thus, by Hilbert Nullstellensatz, $h^{-1} \in A(Y)$.
$\Rightarrow f \in A(Y)$.

• Finally, if ϕ sends an f to 0, then $f|_Y = 0$. Again, by Hilbert Nullstellensatz, $f \in I(Y)$, thus, $f = 0$ in $A(Y)$. $\square$

- To understand $\mathcal{O}_p(Y)$, let us generalize the concept of "fractions" beyond domains:

- Defn: • For a multiplicatively closed $T \subseteq A^*$, define $T^{-1}.A := \{(a,t) \mid a \in A, t \in T\}/\sim$, where $(a,t) \sim (a',t')$ if $at' = a't$.

▷ In our applications $1 \in T$. In that case, $T^{-1}.A$ is a ring extension of A.

- Eg. $(A^*)^{-1}.A = k(A)$ is the <u>fraction field of A</u>.

- For a prime $P \trianglelefteq A$, $(A \setminus P)^{-1}.A =: A_P$ is the <u>localization of A at P</u>.

- We have the natural <u>projective</u> variants:
$T^{-1}.S := \{(s,t) \mid s \in S, t \in T \text{ are homog. \& equi-degree}\}/\sim$

- For a homogeneous prime $\mathfrak{I} \trianglelefteq S$, we have the <u>localization</u> $S_{\mathfrak{I}}$.

- It now easily follows (for an AV Y):
  - $\forall P \in Y, \quad \mathcal{O}_P(Y) \cong A(Y)_{\mathfrak{m}_P}$
  - $K(Y) \cong A(Y)_{\langle 0 \rangle}$
  - $\dim Y = \mathrm{trdeg}_k\, K(Y)$.

- Eg. In $\mathbb{A}^2_k$, $Y = Z(x_2^2 - x_1^3)$ has the fn. field $K(Y) = k(x_1)[x_2]/\langle x_2^2 - x_1^3 \rangle$.

- <u>Abstract remark</u>: The global/local associations $X \mapsto \mathcal{O}(X)$ / $P \mapsto \mathcal{O}_P(X)$ are key in AG to reduce geometry to algebra. This is called a <u>functor</u> & it is a <u>contravariant</u> one (i.e. the arrows, or morphisms, <u>reverse</u> in the two theories!)