- As before, for $Y \subseteq \mathbb{P}^n$, we define the homogeneous ideal $I(Y) := \langle \text{homog. } f \in S \mid \forall P \in Y, f(P) = 0 \rangle$ and the homogeneous coordinate ring $S(Y) := S/I(Y)$.

- So,

$$\begin{array}{rcl}
\mathbb{P}^n_k & \text{- - - - - - - -} & S = k[x_0, .., x_n] \\
Y & \xrightarrow{\quad I \quad} & I(Y) \text{ homog. ideal} \\
\text{closed } Z(T) & \xleftarrow{\quad Z \quad} & T \\
\{\text{closed } Y\} & \xleftrightarrow{1-1} & \{\text{homog. radical } I\} \\
\{PV\ Y\} & \xleftrightarrow{1-1} & \{\text{homog. prime } I\} \\
\{\text{point}\} & \xleftrightarrow{1-1} & \{\text{homog. maximal } I\}
\end{array}$$

- We study the abstract space $\mathbb{P}^n$ by affine coverings!

- Proposition: $\mathbb{P}^n$ has an open covering via $(n+1)$ affine n-spaces.

Pf: • Define $U_i := \{P = (x_0, .., x_n) \in \mathbb{P}^n \mid x_i \neq 0\}$. It is the complement of a hyperplane, hence, open.

• Clearly, $\mathbb{P}^n = \bigcup_{0 \le i \le n} U_i$

• U_i is (homeomorphic to) $\mathbb{A}^n$ via the map $\varphi_i : U_i \to \mathbb{A}^n$; $(x_0, \dots, x_n) \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right)$ with $\frac{x_i}{x_i}$ omitted.

Check that the open sets in U_i & $\mathbb{A}^n$ are in 1-1 correspondence under φ_i. □

▷ This property of $\mathbb{P}^n$ is inherited by all PV & quasi-PV.

- Pf: For a PV Y, write $Y = \bigcup_{i=0}^{n} (Y \cap U_i)$. □

- Eg. In $\mathbb{P}^2_{\mathbb{C}}$, the closed set $Z(x_0^2 - x_1^2)$ has the covering $\{(1, \pm 1, t) \mid t \in \mathbb{C}\} \cup \{(0,0,1)\}$.

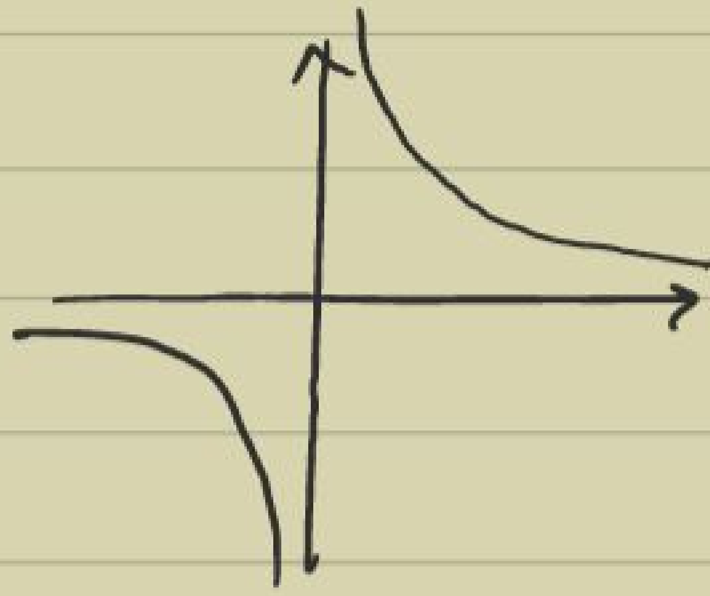
Morphisms

- We want to study maps between AVs X, Y. We need to be extra careful to preserve both the algebraic & geometric properties.

- From topology, we call a map $\varphi : X \to Y$ continuous if $\forall$ open $V \subseteq Y$, $\varphi^{-1}(V)$ is open in X.

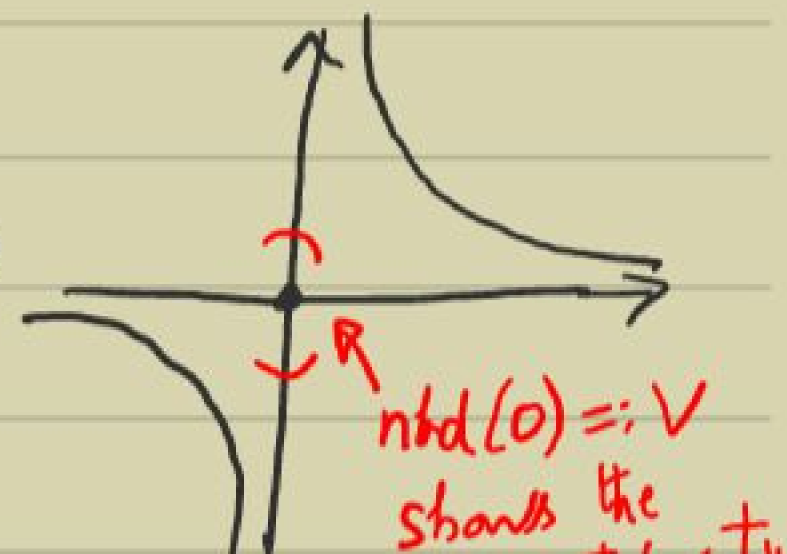
- Eg. $\phi: \mathbb{R}^* \to \mathbb{R}^*$
$x \mapsto 1/x$

is <u>continuous</u>.



But, $\phi: \mathbb{R} \to \mathbb{R}$
$x \mapsto \begin{cases} 1/x, & \text{if } x \neq 0 \\ 0, & \text{else} \end{cases}$

is <u>discontinuous</u>.



▷ Intuitively, in a continuous ϕ: A <u>small</u> change in x leads to a <u>small</u> change in ϕ.

- Also, maps that are <u>locally</u> <u>rational fns.</u> are somehow special.

- Thus, we are led to the following defn, for any quasi-AV $Y \subseteq \mathbb{A}^n$:

Defn: • A fn. $f : Y \to k$ is regular at $P \in Y$ if $\exists$ open nbd. $U \subseteq Y$ $(P \in U)$ s.t. $f = g/h$ on U, for some $g, h \in A$.

• f is regular on Y if it is regular at every pt. of Y.

- We have a similar definition for a quasi-PV $Y \subseteq \mathbb{P}^n$; except we need $g, h \in S_d$, for some $d \geq 0$.

- Eg. In $\mathbb{P}^1_{\mathbb{R}}$, $f := \begin{cases} x_1/x_0, & \text{if } x_0 \neq 0 \\ 0, & \text{else} \end{cases}$

is regular at $(1,0)$ but not at $(0,1)$.

- Eg. $f = (x_1^2 + 1)^{-1}$ is regular on $\mathbb{A}^1_{\mathbb{R}}$ but not on $\mathbb{A}^1_{\mathbb{C}}$!