- An explicit way to compute dim of $Y = Z(\mathfrak{p})$ -

Theorem 1: $\dim Y = \operatorname{trdeg}_k A(Y)$.

- **Defn**: trdeg_k of a domain B is the number of algebraically independent $x_1, \ldots, x_t \in B$ s.t. $k(x_1, \ldots, x_t) \subseteq k(B)$ is a finite (algebraic) extension of fields.

- Eg. $\operatorname{trdeg}_k k[x,y]/\langle y^2 - x^3 \rangle = 1$.

  Because, $k(x) \subset k(x)[y]/\langle y^2 - x^3 \rangle$ is a quadratic extn.

- Pf-sketch (Thm 1): • Say, $d := \dim Y$ & the max. chain of AV in Y is: $Z(\mathfrak{p}_0) \subsetneq Z(\mathfrak{p}_1) \subsetneq \cdots \subsetneq Z(\mathfrak{p}_d)$.

  • Thus, we have an opposite chain of primes: $\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_d$ (of A).

  • This gives us a chain of fields (in $k(A(Y))$): $k(A/\mathfrak{p}_0) \subsetneq k(A/\mathfrak{p}_1) \subsetneq \cdots \subsetneq k(A/\mathfrak{p}_d)$.

- As, this is the max. possible such chain of fields in $k(A/\mathfrak{p})$, each field extn. above contributes a one to the trdeg.
$\Rightarrow \operatorname{trdeg}_k k(A/\mathfrak{p}) = d$. $\square$

Projective varieties

- We now study homogeneous polynomials, i.e. equi-degree monomials, eg. $x^2 - xy + y^2$.
- Note that any root $(x_1, \ldots, x_n)$ of a homogeneous polynomial gives rise to roots of the kind $(\lambda x_1, \ldots, \lambda x_n)$ for $\lambda \in k$.
  Why not discount these extra roots?

Defn: • Let us consider the equivalence relation in $(\mathbb{A}_k^{n+1})^*$: $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$ if $\exists \lambda \in k^*$, $(\lambda a_0, \ldots, \lambda a_n) = (b_0, \ldots, b_n)$.
• Define $\mathbb{P}_k^n := (\mathbb{A}_k^{n+1})^* / \sim$ as the projective n-space over k.

- For $p \in \mathbb{P}^n$, any $(a_0, \dots, a_n)$ in the equivalence class of p is called a <u>set of homogeneous coordinates for p</u>. This is sometimes denoted by $(a_0 : a_1 : \dots : a_n)$.

— eg: In $\mathbb{P}^1_{\mathbb{C}}$ the pts. $(2, 1-i)$ & $(1+i, 1)$ are the <u>same</u>. But, $(2, 1-i)$ & $(1, 1-i)$ are <u>not</u>.

— Intuitively, $\mathbb{P}^n$ is the space of <u>all</u> lines passing through a <u>fixed</u> point in $\mathbb{A}^{n+1}$.

— What is the algebraic analogue of $\mathbb{P}^n_k$?

— Let $S := k[x_0, \dots, x_n]$. Define a <u>grading</u> $S = \bigoplus_{d \geq 0} S_d$, where $\underline{S_d}$ is the k-linear span of all monomials of deg d in S.

— I.e. grading is essentially a decomposition of a poly. in S, into homogeneous parts.

$\triangleright$ $\forall d,e \in \mathbb{N}$, $S_d \cdot S_e \subseteq S_{d+e}$.

- <u>Defn</u>: An ideal $I \trianglelefteq S$ is called <u>homogeneous</u> if I has a set of homogeneous generators.

- <u>Proposition</u>: (1) I is homogeneous iff $I = \bigoplus_{d \geq 0} (I \cap S_d)$.
  (2) Sum, product, intersection, radical of homogeneous ideals is homogeneous.

<u>Pf</u>: (Exercise) $\square$

- Eg. $\langle x_0, x_1^2 \rangle \overset{+}{\underset{\cap}{\cdot}} \langle x_0 x_1 \rangle = \begin{matrix} \langle x_0, x_1^2 \rangle \\ \langle x_0^2 x_1, x_0 x_1^3 \rangle \\ \langle x_0 x_1 \rangle \end{matrix}$

- Note that if $f \in S_d$ then, $f(\lambda \bar{x}) = \lambda^d \cdot f(\bar{x})$. This motivates the definition:

- <u>Defn</u>: • If $T \subseteq S$ has homogeneous polys. then, $\underline{Z(T)} := \{ p \in \mathbb{P}^n \mid \forall f \in T, f(p) = 0 \}$.
  • Similarly, for a homogeneous ideal $I \trianglelefteq S$, we have $\underline{Z(I)}$.

- Eg. In $\mathbb{P}^1_{\mathbb{C}}$, $Z(x_0^2) = \{(0,1)\}$, &
$Z(x_0^2 - x_1^2) = \{(1, \pm 1)\}$.

- Defn: • A $Y \subseteq \mathbb{P}^n$ is closed (or algebraic) if $\exists$ homogeneous $T \subseteq S$ s.t. $Y = Z(T)$.
• For closed Y, $\mathbb{P}^n \setminus Y$ is open.

▷ Again, the family of open subsets in $\mathbb{P}^n$ form the Zariski topology on $\mathbb{P}^n$.

- Defn: • A projective variety is an irreducible closed subset in $\mathbb{P}^n$.
• An open subset of a PV is called a quasi-projective variety.
• Their dimension as before.

- Eg. In $\mathbb{P}^1_{\mathbb{C}}$, $Z(x_0^2)$ is a PV, but $Z(x_0^2 - x_1^2)$ is not. $\dim Z(x_0^2) = 0$.
In $\mathbb{P}^2_{\mathbb{C}}$, $Z(x_0^2) \setminus Z(x_0^2, x_1)$ is a quasi-projective curve!