- Let us go back to the $Z(\cdot)$ & $I(\cdot)$ operators. Assume $k$ algebraically closed.
- <u>Defn</u>: <u>Radical</u> of an ideal $\mathcal{I}$:

$$\sqrt{\mathcal{I}} := \{ f \in A \mid \exists i \in \mathbb{N},\ f^i \in A \}.$$

- <u>Eg.</u> $\sqrt{\langle x_1^2 x_2 \rangle} = \langle x_1 x_2 \rangle$.

- $\mathcal{I}$ is called a radical ideal if $\mathcal{I} = \sqrt{\mathcal{I}}$.

- <u>Proposition</u>: For $\mathcal{I} \trianglelefteq A$, $I \circ Z(\mathcal{I}) = \sqrt{\mathcal{I}}$.

<u>Pf</u>:
- Clearly, $\sqrt{\mathcal{I}} \subseteq I \circ Z(\mathcal{I})$.
- Conversely, let $g \in I \circ Z(\mathcal{I})$

$\Rightarrow g$ vanishes on $Z(\mathcal{I})$

$\Rightarrow Z(\langle 1 - gy \rangle + \mathcal{I}) = \phi$, where $A' := A[y]$.

- So, by Hilbert Nullstellensatz (weak):

$$1 \in \langle 1 - gy \rangle + \mathcal{I}.$$

$\Rightarrow \exists a_0, \ldots, a_m \in A'$ s.t.

$$1 = a_0(1 - gy) + a_1 f_1 + \cdots + a_m f_m$$

- At $y = \bar{g}^{-1}$ (seen as an element in $k(\mathbb{A}^1)$), we have $1 = \sum_{i=1}^{m} a_i(\bar{x}, \bar{g}^{-1}) \cdot f_i(\bar{x})$

$\Rightarrow \exists j \in \mathbb{N}$, $g^j = \sum_{i=1}^{m} b_i f_i$, for some $b_i \in A$.

$\Rightarrow g \in \sqrt{I}$. $\square$

▷ Thus, $\mathbb{A}_k^n \dashleftarrow\dashrightarrow A$

$\{\text{algebraic } Y\} \xleftrightarrow{1\text{-}1} \{\text{radical } I\}$

▷ Moreover, $\{\text{affine variety } Y\} \xleftrightarrow{1\text{-}1} \{\text{prime } I\}$.

- Pf: • Let $I$ be radical & $Y := Z(I)$. Thus, $I(Y) = I \circ (Z(I)) = \sqrt{I} = I$.

• If $I$ is not prime, then $\exists f, g \in A \setminus I$ s.t. $fg \in I$.

• This leads to the decomposition:

$$Z(I) = Z(\langle f \rangle + I) \cup Z(\langle g \rangle + I)$$

showing that $Y$ cannot be an affine variety.

- Conversely, assume that $I$ is prime. A similar proof shows that $Z(I)$ is an AV. $\square$

▷ Finally, $\{\text{pt. in } Y\} \xleftrightarrow{1\text{-}1} \{\text{max. ideal } I\}$.

- **Defn:** For an algebraic $Y \subseteq \mathbb{A}^n$, we define the <u>coordinate ring</u> $A(Y) := A/I(Y)$.

  - $A(Y)$ is thought of as the ring of fns. on $Y$.
  - Eg. For an irred. $f \in A$, denote $Y := Z(f)$. Then, $A(Y) = A/I(Y) = A/\langle f \rangle$ are the <u>fns.</u> defined <u>on the zeros of $f$</u>!

  - Intuitively, we know $\dim(\mathbb{R}^1) = 1$, $\dim(\mathbb{R}^2) = 2$, ... Is there a geometric defn. of dim, for <u>any</u> AV?

- **Defn:** The <u>dim</u> of an AV $Y$ is the max. $n \in \mathbb{N}$ s.t. $\exists$ a chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ of affine varieties in $Y$.

- e.g. in $\mathbb{A}^1$: $\{0\} \subsetneq \mathbb{A}^1 \Rightarrow \dim \mathbb{A}^1 \geq 1$

  in $\mathbb{A}^2$: $\{0\} \subsetneq \mathbb{A}^1 \subsetneq \mathbb{A}^2 \Rightarrow \dim \mathbb{A}^2 \geq 2$.

  <u>Exercise</u>: Prove the equality!

▷ An AV of $\dim = 0$ is simply a point.

- <u>Defn</u>: An AV of $\dim = 1$ is called an (affine) <u>curve</u>.

- <u>e.g.</u> For an irred., non-constant $f \in A = k[x_1, x_2]$ denote $Y = Z(f)$. The <u>only</u> possible chain in $Y$ is: $\{P\} \subsetneq Y$, for $P \in Y$. Thus, $\dim Y = 1$ and its a curve.

- <u>e.g.</u> In $\mathbb{A}^3_k$: $Y = Z(x_1, x_2)$ is also a curve!