Grid Graph Reachability in UL

By
Massand Sagar Sunil

Graph reachability characterizes various classes. In particular, the directed graph reachability is known to be NL-complete while undirected graph reachability is known to be in L.

Our job:

Show Directed Planar Reachability in UL

Proof Idea:

•Reduce Directed Planar Reachability to Grid Graph Reachability.

•Show that Grid Graph Reachability is in UL.

a. Reduce grid graph reachability to min-unique reachability.

b. Show that min-unique reachability is in UL.

Grid Graph:

NxN graph such that the following holds:

If there is an edge from (x_1,y_1) to (x_2,y_2),

a. $x_1=x_2$ and $|y_1-y_2|=1$
b. $y_1=y_2$ and $|x_1-x_2|=1$.

Min-unique graph :

- Directed Graph with weighted edges.

- Weight of a path is sum of weights of edges.

- If there is a path from u to v, then there is a unique minimum weight path from u to v.

Theorem (Reinhardt and Allender [3]). Let G be a class of graphs and let H = (V,E) ∈ G. If there is a polynomially-bounded log-space computable function f that on input H outputs a weighted graph f(H) so that

1. f(H) is min-unique, and
2. H has an st-path if and only if f(H) has an st-path. Then the st-connectivity problem for G is in UL

We first show an easier result.

Layered Grid Graph reachability is in UL.

Layered Grid Graph:

NxN graph such that the following holds:

If there is an edge (x_1,y_1) to (x_2,y_2),

a. $x_1=x_2$ and $y_2-y_1=1$.
b. $y_1=y_2$ and $x_2-x_1=1$.

Layered grid graph-**Weight function**:

$W(e)=x$ if edge from $(x,y) \rightarrow (x,y+1)$
$W(e)=0$ if edge from $(x,y) \rightarrow (x+1,y)$.

Note the following:

The path is simple, due to the restrictions on the graph.

Weight cannot be same.

We now move to the case of the general grid graph reachability problem.

Determining if d(v) ≤ k or not

Input: (G, v, k, c_k, $\sum_k$)

Output: true if d(v)≤k else false

```
Initialize count ← 0; sum ← 0; path.to.v ← false
foreach x ϵ V do
        Nondeterministically guess if d(x) ≤ k
        if guess is Yes then
                Guess a path of length l ≤ k from s to x
                if guess is correct then
                        Set count ←  count + 1
                        Set sum ←   sum + l
                        if x = v then set path.to.v ← true
                Else
                        return "?"
                End
        End
End

if count = c_k and sum = ∑_k then
        return path.to.v
Else
        return "?"
end
```

Computing ck and $\sum_k$

Input: $(G, k, c_{k-1}, \sum_{k-1})$

Output: $c_k, \sum_k$

Initialize $c_k \leftarrow c_{k-1}$ and $\sum_k \leftarrow \sum_{k-1}$

foreach $v \in V$ do

    if $\neg(d(v) \leq k - 1)$ then

        foreach x such that $(x, v) \in E$ do

            if $d(x) \leq k - 1$ then

                Set $c_k \leftarrow c_{k+1}$

                Set $\sum_k \leftarrow \sum_k + k$

            end

        end

    end

end

return c_k and $\sum_k$

Weight function

W(e)= n^4 if e is an edge from $(x_1,y) \rightarrow (x_2,y)$ where $|x_2-x_1|=1$

$x+n^4$ if e is an edge from $(x,y) \rightarrow (x,y+1)$

$-x+n^4$ if e is an edge from $(x,y) \rightarrow (x,y-1)$.

Positive weight edges imply simple path

Uniqueness of path:

Weight of path can be divided into two components $C(p) + n^4D(p)$.

D(p) would denote the number of edges on the path.

Weight of a cycle

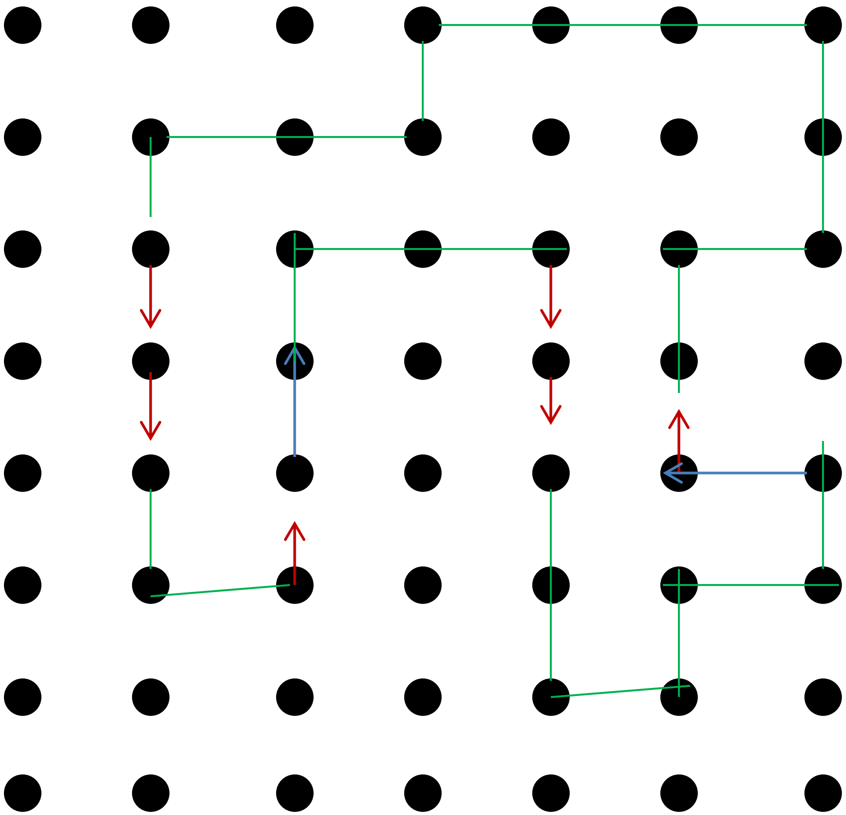
For a simple cycle, we show the following:

The weight of the path is:

A(C) in case of a counter-clockwise path

-A(C) in case of a clockwise path

A(C) is the number of blocks enclosed by the simple cycle.



References:

[1]-Chris Bourke, Raghunath Tewari and N.V. Vinodchandran, Directed Planar Reachability is in Unambiguous Logspace.

[2]-The Directed Planar Reachability by Eric Allender, Samir Datta, Sambhudha Roy

[3]- Klaus Reinhardt and Eric Allender, Making nondeterminism unambiguous, SIAM Journal of Computing, 29 (2000), pp. 1118–1131.