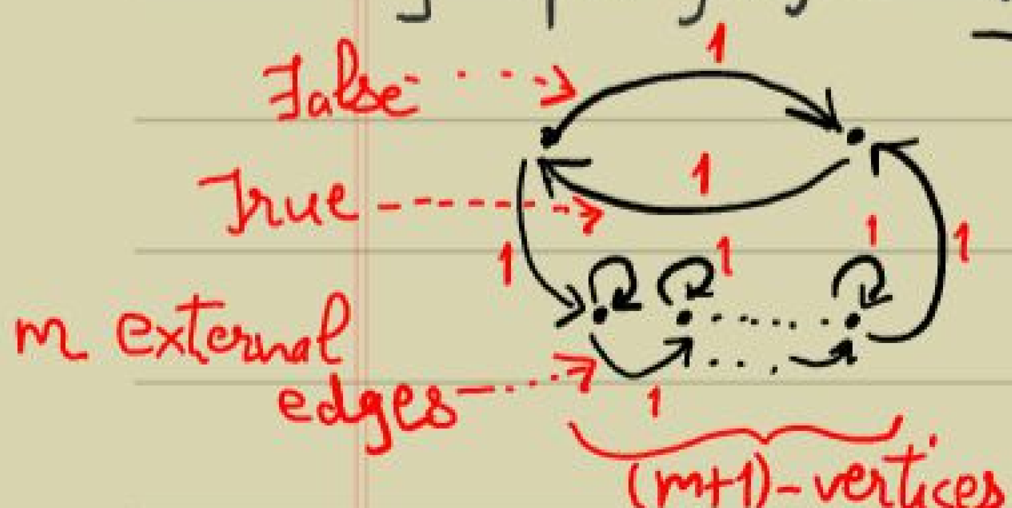
Theorem: per for 0/1 matrices is #P-hard.

Proof:

- We will relate #3SAT to permanent.
- Let φ be a 3CNF formula with m clauses & n variables.

  Wlog each clause has exactly three literals.
- We will construct a graph A s.t. the sum of its weighted cycle-covers $=: \mathrm{per}(A) = \#(\text{sat. assgns. of } \varphi)$.

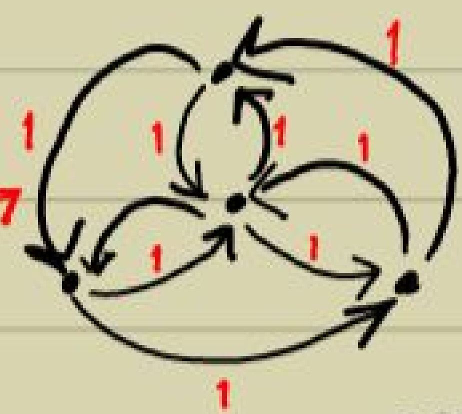
- Each variable x_i is converted to a graph gadget V_i:



- The cycle-cover using the external edges signifies $x_i = T$ while the other signifies $x_i = F$.

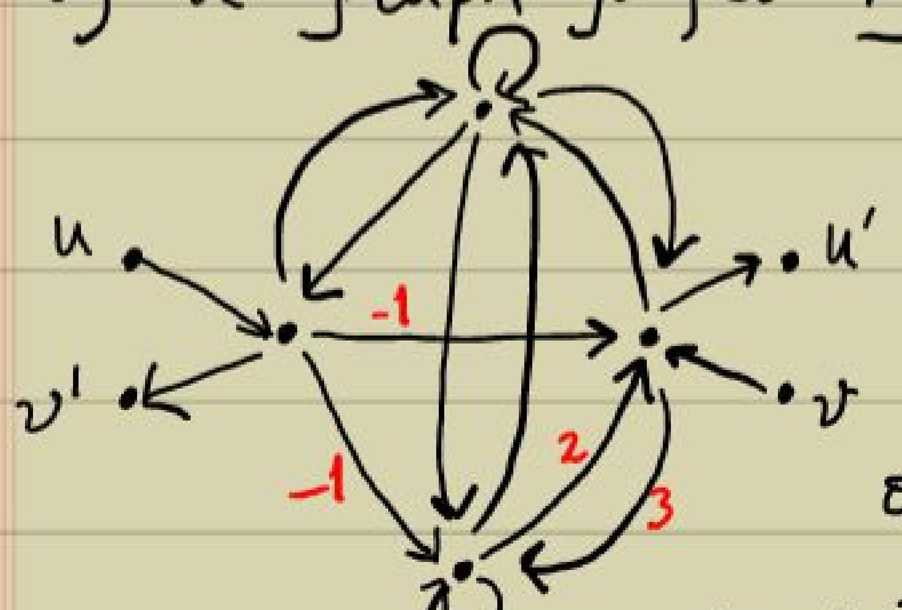
- Each clause F_j is converted to a graph gadget $\underline{C_j}$:

3external edges



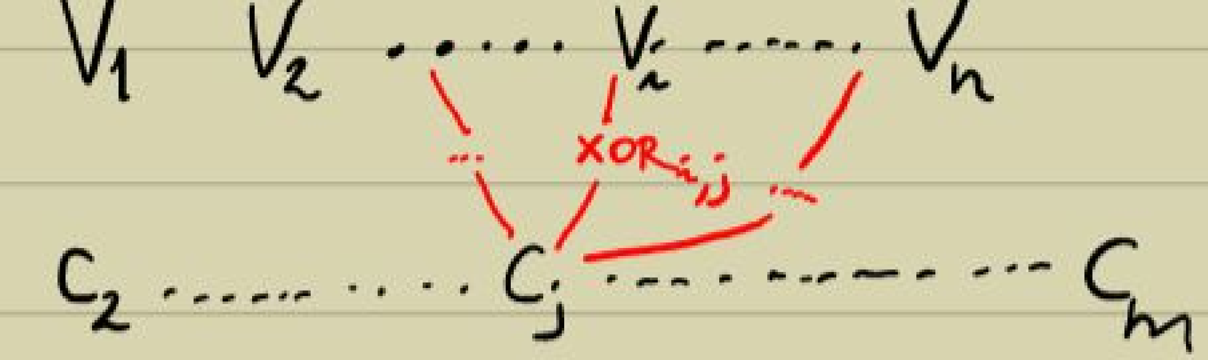
- There are 3 cycle-covers each of $wt=1$, each corresponding to a dropped external edge. The latter signifies that the corresponding literal in F_j is True.

- If x_i is in F_j then the j-th external edge (u, u') of V_i is connected to the x_i-external-edge (v, v') of C_j by a graph gadget $\underline{XOR_{i,j}}$:



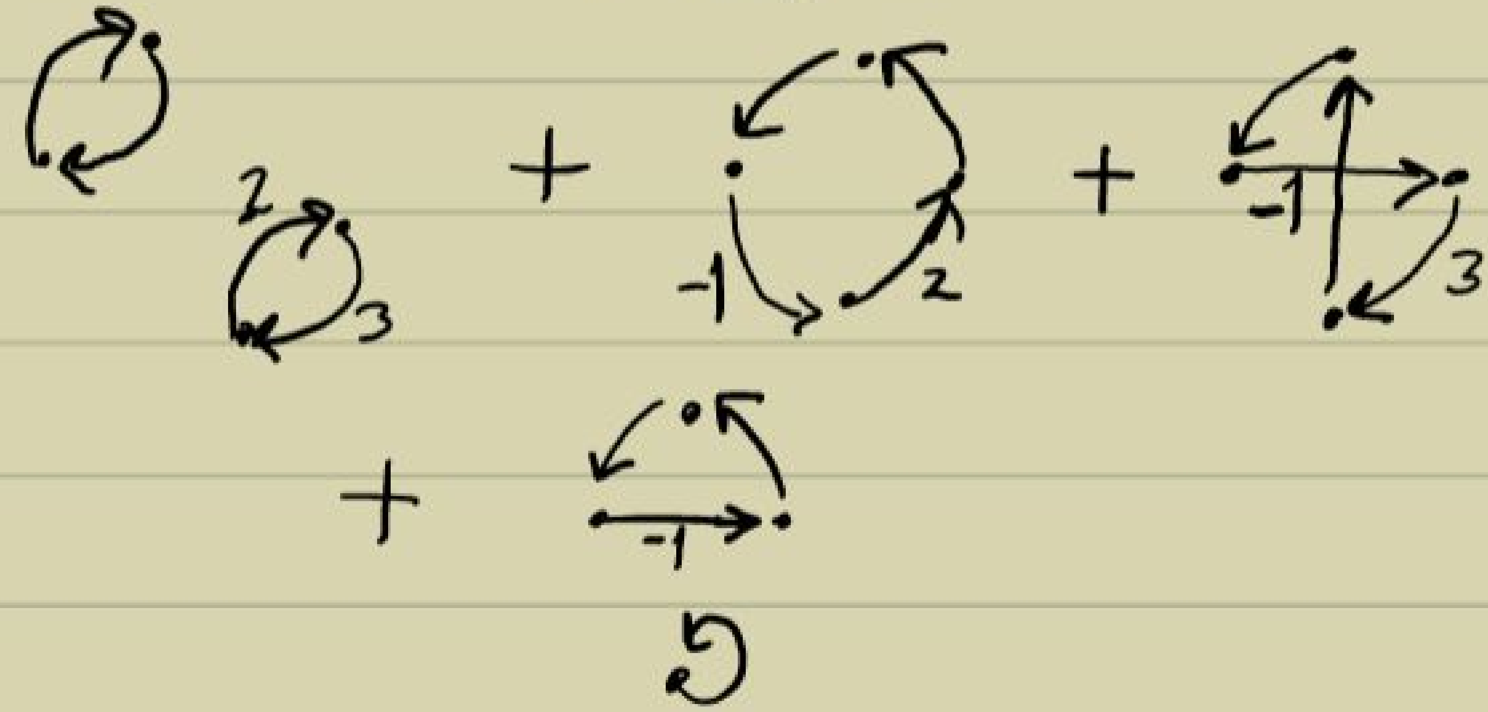
- If $\bar{x}_i$ is in F_j then the False edge (u, u') of V_i is connected to x_i-external-edge (v, v') of C_j by $XOR_{\bar{i},j}$.

- Finally, the graph A' is:



<u>Claim</u>: The cycle covers of XOR sum to weight 0.

<u>Pf</u>: • Cycle-covers in the gadget $XOR \setminus \{u, u', v, v'\}$:



$= 6-2-3-1 = 0.$ $\square$

- Thus, the cycle-covers of A' involving $XOR_{i,j}$ are those that corresponds to <u>exactly one</u> of the paths (u,u') or (v,v').

- Consider such a cycle-cover $\mathcal{C}$ of A':

For every C_j, there is an external edge, say x_i, <u>not</u> appearing in $\mathcal{C}$.

$XOR_{i,j}$ ensures that the j-th external edge in V_i appears in $\mathcal{C}$.

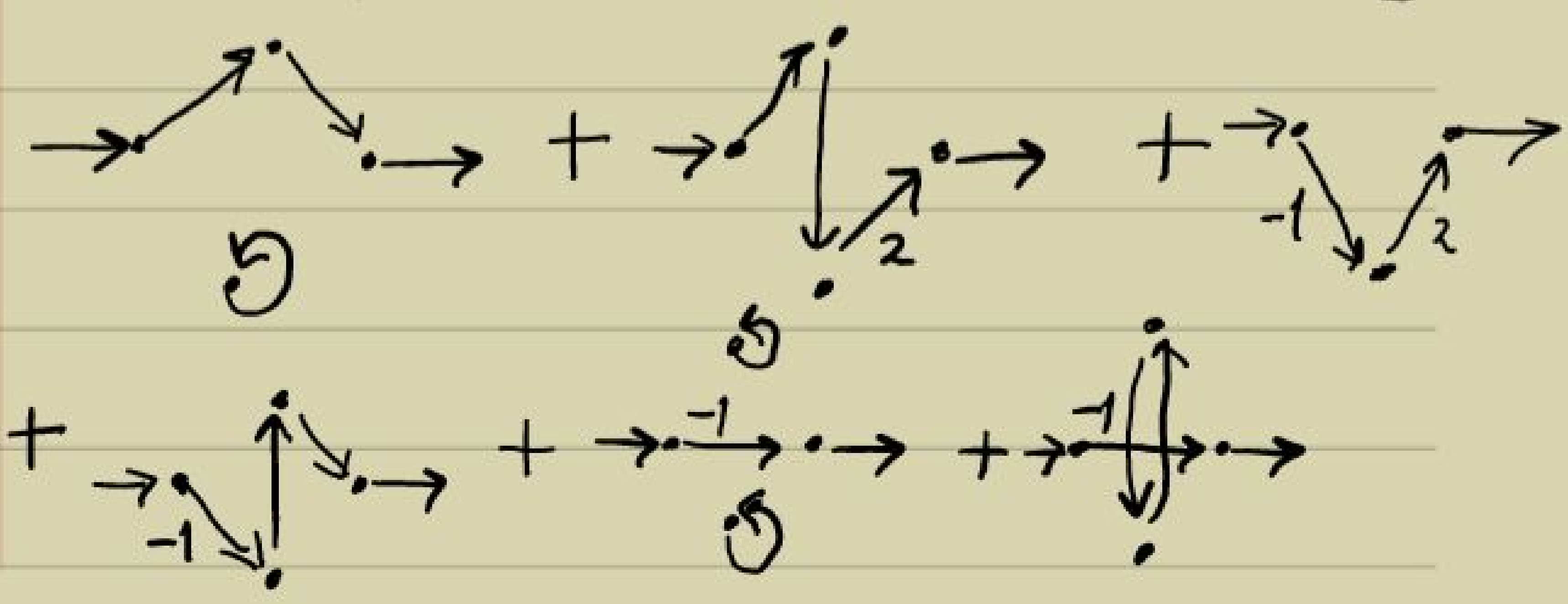
$\Rightarrow \mathcal{C}$ uses the external cycle-cover of V_i

$\Rightarrow x_i = \text{True}$ is captured!

$\Rightarrow \mathcal{C}$ corr. to a sat. assgn. of ϕ.

<u>Claim:</u> The contribution of $XOR_{i,j} \setminus \{v, v'\}$ to the weight is $-2 \neq 0$. ← char. of the field <u>not</u> 2.

<u>Pf:</u> • Paths from u to u':



$= 1+2-2-1-1-1 = -2.$ $\square$

$\Rightarrow$ For a satisfying assignment s of φ: $\displaystyle\sum_{\substack{\text{cycle-cover } C \\ \text{of } A' \text{ corr. to } s}} wt(C) = (-2)^{3m}$.

the #clauses is m & each has exactly 3 literals

$$\Rightarrow per(A') = \sum_{\substack{\text{cycle-cover} \\ C \text{ of } A'}} wt(C)$$

$$= (-2)^{3m} \cdot \#(\text{sat. assgns. of } \varphi).$$

- Finally, we want to reduce A' to a 0/1 matrix.

(i) $|per(A')| < 2^{3m+n+1} =: 2^N$.

So, we can replace a negative entry $-k$ in A' by $(2^N - k)$; get a matrix A'' & consider $per(A'') \pmod{2^N}$.

(ii) A'' has non-negative, but large, entries.

Replace a weighted edge $u \xrightarrow{k} u'$ by a gadget of sequential & parallel weight ≤ 2 paths.

e.g. $u \xrightarrow{5} u'$ by:

• This gadget is of size $O(\lg k)$.

▷ Finally, we get a 0/1 matrix A, of dim. $O(mn(m+n))$ s.t.

$$\mathrm{per}(A) \bmod 2^N = (-2)^{3m} \cdot \#\phi .$$

$\Rightarrow \#SAT \in FP^{\text{per on } 0/1}$