#P-completeness

- We call a fn. $f : \{0,1\}^* \to \mathbb{N}$ **#P-complete** if $f \in \#P$ & $\#P \subseteq FP^f$.
  functions computable by a poly-time TM using f as an oracle.

▷ If an $f \in \#P$ is #P-complete, then $FP = \#P$.

Theorem: #SAT is #P-complete.

Proof:
- We know that $\#SAT \in \#P$.
- Since the computation of a polytime NDTM can be encoded in a boolean formula, we deduce:

  For any $f \in \#P$ with $f(x) = \#$(accepting paths of an NDTM M on x), we get a formula $\varphi_{M,x}$ s.t. $f(x) = \#$(sat. assgn. of $\varphi_{M,x}$).

$\Rightarrow f \in FP^{\#SAT}$

$\Rightarrow \#P \subseteq FP^{\#SAT}$. $\square$

An algebraic #P-complete problem

- For a matrix $A \in \mathbb{F}^{n \times n}$, <u>permanent</u> is defined as:

$$\underline{per(A)} := \sum_{\sigma \in Symm_n} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

Symm$_n$ is the group of $n!$ permutations of $[n]$.

- Notice the similarity with

$$det(A) = \sum_{\sigma \in Symm_n} sgn(\sigma) \cdot \prod_{i \in [n]} A_{i,\sigma(i)}$$

- We will see that per$(\cdot)$ is one of the hardest problems in #P !

<u>Lemma 1:</u> Per, for 0/1 matrices, is in #P.

<u>Proof:</u>
- Let $A \in \{0,1\}^{n \times n}$.
- $per(A) = \#\{\sigma \in Symm_n \mid \prod_{i=1}^{n} A_{i,\sigma(i)} = 1\}$.

$\Rightarrow$ per(A) is the #(acc. paths. of the NDTM that given A guess the σ. $\square$

- Permanent has several known applications in computational physics & probability.

Graph interpretation of per

<u>Defn</u>: • Given $A \in \mathbb{F}^{n\times n}$, view it as the adjacency matrix of a <u>weighted digraph</u> on n vertices.

• A <u>cycle-cover</u> C of A is a subgraph having n vertices, each vertex of in-deg = out-deg = 1.

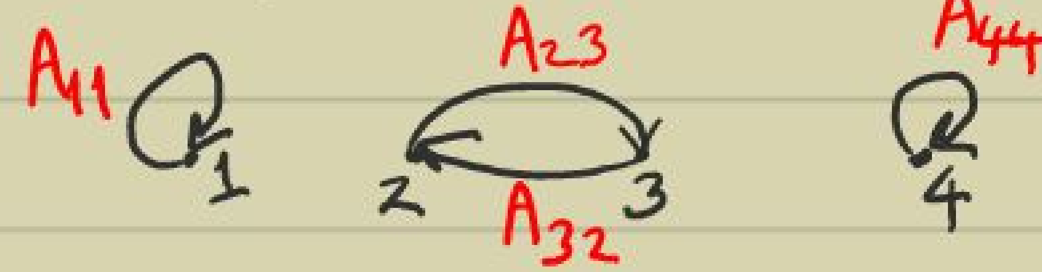
• Thus, C is a union of disjoint cycles.

• <u>$wt(C)$</u> := product of weights of edges in C.

$$\triangleright \quad per(A) = \sum_{C \in \text{cycle-cover}(A)} wt(C) .$$

Pf. sketch: • Say, a monomial $A_{11}A_{23}A_{32}A_{44}$ appears nontrivially in per(A).

$\Rightarrow$ The digraph G, corr. to A, has the cycle-cover:



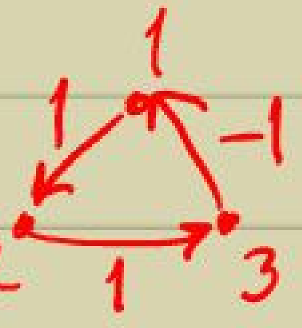
• Conversely, it can be seen that the wt. of a cycle-cover of G corresponds to a monomial of per(A). $\square$

Eg. $A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$; $G =$

Cycle-covers of A:

wt = 1

&



wt = -1

; per(A) = 1 − 1 = 0.