- Thus, the question $NP \overset{?}{\neq} PSPACE$ is asking whether "games" are harder than "puzzles"!

- Using the proof of the previous theorem we can also show Npspace = Pspace.

<u>Theorem</u> (Savitch 1970): $\text{Nspace}(S) \subseteq \text{Space}(S^2)$.

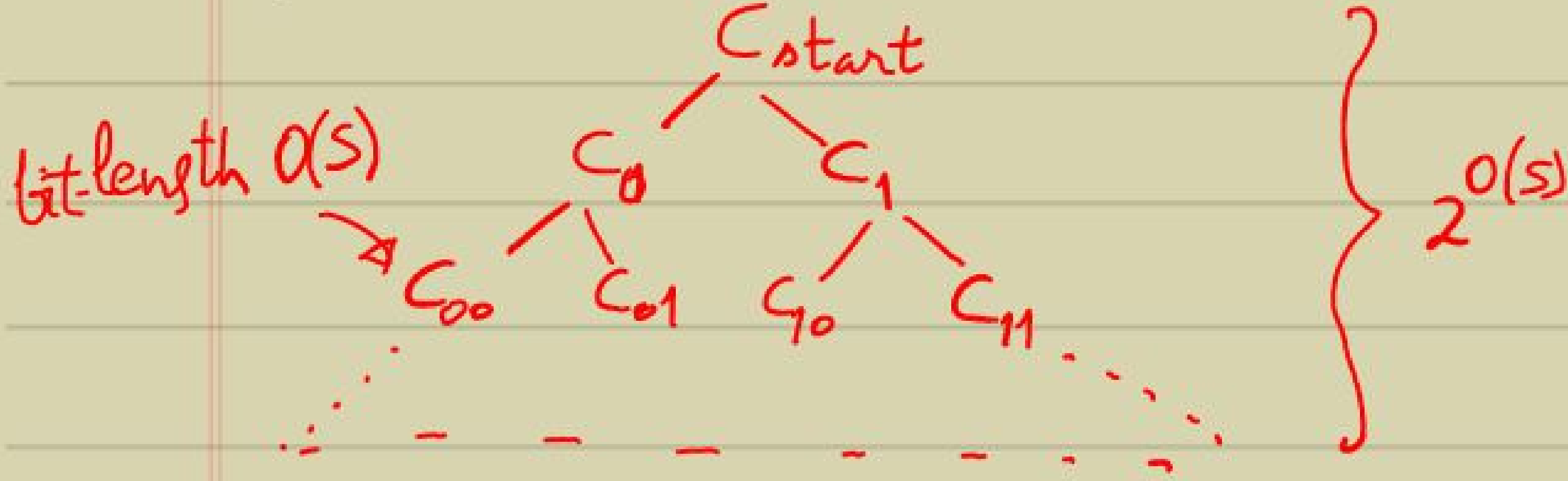
<u>Proof</u>:

- Let M be an $S(n)$-space NDTM deciding $L \in \text{Nspace}(S)$.
- As in the proof of Lemma 2, we design a QBF: $\psi_{M,x} = \psi_{d \cdot S(n)}(\underset{\text{with } x}{C_{start}}, C_{accept})$,

with the modification that $\varphi_{M,x}(C, C')$ captures <u>two</u> possible transition steps instead of a unique one.

- $\psi_{M,x}$ remains a QBF of size $O(S^2)$.
- So, easily checkable in $\text{Space}(S^2)$. $\square$

- Another way to interpret the previous result is through the configurations tree of the NDTM M:



▷ Each configuration requires $O(S)$ space & specifying a location in the tree requires $\lg 2^{O(S)}$ space.

$\Rightarrow$ <u>Reachability</u> $C_{start} \rightsquigarrow C_{stop}$ can be checked in $O(S^2)$ space.

(Alternately, do it in $2^{O(S)}$ time & space!)

- Reachability also plays an important role in the "small" classes NL, L, ...

NL-completeness

- By the previous discussion
$$\mathbb{L} \subseteq NL \subseteq P.$$

- For a meaningful notion of "hardness" in NL we need $\mathbb{L}$-reductions.

Defn: • We call $f : \{0,1\}^* \to \{0,1\}^*$ implicitly $\mathbb{L}$-computable if
$$L_f := \{(x,i) \mid f(x)_i = 1\}, \ \&$$
$$L'_f := \{(x,k) \mid |f(x)| \geq k\}$$
are in $\mathbb{L}$.

• We call $A \leq_{\mathbb{L}} B$ if $\exists$ implicitly $\mathbb{L}$-computable f s.t. $\forall x \in \{0,1\}^*$,
$$x \in A \text{ iff } f(x) \in B.$$

• We call B NL-complete if
$$B \in NL, \ \&$$
$$\forall A \in NL, \ A \leq_{\mathbb{L}} B.$$

Proposition: (i) $A \leq_{\mathbb{L}} B \leq_{\mathbb{L}} C \Rightarrow A \leq_{\mathbb{L}} C$.

(ii) $A \leq_{\mathbb{L}} B$ & $B \in \mathbb{L} \Rightarrow A \in \mathbb{L}$.

Proof:

(i) Let f, g be the two implicitly $\mathbb{L}$-computable fns. Observe that $g \circ f$ is also implicitly $\mathbb{L}$-computable.

(ii) Similar. $\square$

- We now see an NL-complete problem.

Defn: Path $:= \{(G, s, t) \mid \exists$ directed path $s \rightsquigarrow t$ in the directed graph $G\}$.

Lemma 1: Path $\in$ NL.

Pf:
- Let (G, s, t) be an input instance.
- Each vertex in G can be identified in $\lg|G|$ space.
- Thus, an NDTM M that simulates a walk in G (with origin s & accepts on reaching t),

has space complexity $O(\lg n)$.

$\Rightarrow$ Path $\in$ NL. $\square$

Lemma 2: $\forall A \in NL$, $A \leq_{L} Path$.

Proof:

- Let M be an NDTM deciding A in space $\leq d \cdot \log n$.
- We consider an f that maps an input x of A to $f(x) = (G, s, t)$ where G is the configuration graph of $M(x)$:
  (1) the vertices of G are all the configs. of $M(x)$,
  (2) s & t are the "start" & "accept" configs. respectively,
  (3) the edge (C, C') is there iff $C \rightarrow C'$ is a valid transition step of $M(x)$.

- Obviously, $x \in A$ iff $f(x) \in Path$.

- More importantly, f is implicitly $\mathbb{L}$-computable:
  (1) the list of vertices is $\mathbb{L}$-computable,
  (2) given (C,C') we can check whether its a valid transition of $M(x)$ in $O(|C|+|C'|) = O(\lg|x|)$ space.
  (Basically, scan C,C' & refer the δ-fn. of the TM M.) $\square$

<u>Theorem</u>: Path is NL-complete.

<u>Open</u>: $NL \neq \mathbb{L}$?

<u>Corollary</u>: $\overline{\text{Path}}$ is coNL-complete.

<u>Pf</u>: ◦ Use the same f as before. $\square$

<u>Qn</u>: Is $NL = coNL$?