Proof: • Observe that a TM using $f(n)$ space can have $O(2^{f(n)})$ configurations.
(More than this makes the computation cyclic.)

• So, $2^{f(n)}$ upper bounds the time complexity. $\square$

Pspace completeness

Defn: A language B is Pspace-complete if
- $B \in$ Pspace, and
- $\forall A \in$ Pspace, $A \leq_p B$.

- We saw that $\exists$-quantifiers gave us an NP-complete problem.
What happens if we also use $\forall$?

Defn: $\text{TQBF} := \{ \overbrace{Q_1 x_1 \cdots Q_n x_n \, \phi(\bar{x})}^{\text{quantified boolean formula}} \mid Q_i\text{'s are}$ quantifiers, ϕ is a boolean formula in $x_1, \ldots, x_n$ & $Q_1 x_1 \ldots Q_n x_n \, \Phi(\bar{x})$ is true$\}$.

Lemma 1: TQBF $\in$ Pspace.

Proof:

- Let $\psi := Q_1 x_1 \ldots Q_n x_n \, \Phi(\bar{x})$ be a QBF with $|\Phi| =: m$.
- We can check its truth by the following recursive algorithm:

ψ is true iff

$$Q_1 = \exists \;\&\; (Q_2 x_2 \ldots Q_n x_n \, \Phi(0, x_2, \ldots, x_n) \text{ or } Q_2 x_2 \ldots Q_n x_n \, \Phi(1, x_2, \ldots, x_n))$$

$$Q_2 = \forall \;\&\; (Q_2 x_2 \ldots Q_n x_n \, \Phi(0, x_2, \ldots, x_n) \;\&\; Q_2 x_2 \ldots Q_n x_n \, \Phi(1, x_2, \ldots, x_n)).$$

- Its space complexity $s(n,m)$ is given by:

$$s(n,m) \leq \underbrace{s(n-1,m)} + O(m).$$ ← we reuse space, & not double it

Improve to $O(n+m)$ → $\Rightarrow s(n,m) = O(nm)$.

$\Rightarrow$ TQBF $\in$ Pspace. $\square$

Lemma 2: $\forall L \in$ Pspace, $L \leq_p$ TQBF.

Proof:

- Let M be an $S(n)$-space TM deciding $L \in$ Pspace.
- We will construct a QBF $\psi_{M,x}$, of size $O(S(n)^2)$, whose truth depends on M accepting x.

- By Cook-Levin reduction, we have a formula $\varphi_{M,x}(C, C')$, of size $O(S)$, that is true iff $C \to C'$ is a valid transition step of configurations of M on x.
- Now, M on input x ($|x| =: n$) can have at most $2^{d \cdot S(n)}$ distinct configurations (for some constant d).
- It is possible to design a QBF $\psi_i(C, C')$ that captures whether C to C' is reachable in $\leq 2^i$ transition steps of M:

Recursive defn

$$\psi_i(C, C') := \exists C'' \left(\psi_{i-1}(C, C'') \wedge \psi_{i-1}(C'', C') \right).$$

- This gives us the QBF
  $$\psi_{M,x} := \psi_{d \cdot S(n)}(C_{\substack{start \\ with\ x}}, C_{accept}).$$

- Clearly, this is of size $2^{d \cdot S(n)}$.
  How do we improve?

<u>Idea</u>: "Reuse" space for ψ_{i-1}!

Re-define,

- $$\psi_i(C, C') := \exists C'' \ \forall D_1 \ \forall D_2 \ \left[(D_1 = C \wedge D_2 = C'') \vee (D_1 = C'' \wedge D_2 = C') \Rightarrow \psi_{i-1}(D_1, D_2) \right].$$

- Notice that now we get a $\psi_{M,x}$ of size $(\lg 2^{d \cdot S(n)}) \cdot O(S(n)) = O(S(n)^2)$.
- Also, M accepts x iff $\psi_{M,x} \in TQBF$.

$\Rightarrow L \leq_p TQBF$. $\square$

- Lemma 1 & 2 prove:

<u>Theorem</u> (Meyer & Stockmeyer, 1972):

TQBF is Pspace-complete.

<u>The QBF game</u>

- The truth of a QBF $\psi :=$

  $\exists x_1 \forall x_2 \exists x_3 \ldots \exists x_{2n-1} \forall x_{2n} \; \phi(x_1, \ldots, x_{2n})$

  can be interpreted as a 2-player game.

- Say, Player-1 picks values for $x_1, x_3, \ldots$
  & Player-2 " " " $x_2, x_4, \ldots$

- We declare Player-1 the <u>winner</u> iff ϕ is true in the end.

  $\Rightarrow$ Deciding $\psi \overset{?}{\in}$ TQBF signifies whether there <u>is</u> a winning strategy for Player-1!

  $\Rightarrow$ Suggests the Pspace-hardness of many board games, eg. Chess_n, Go_n, Checkers_n.