Formalizing Problems & Inputs

- A problem for us would always mean computing a fn. f whose input & output are finite boolean strings.

$$f : \{0,1\}^* \rightarrow \{0,1\}^* .$$

- This captures almost all math. fns. as we can express the objects "integer x", "graph G", "vector v", "matrix A", etc. in bits!

- Usually, problems have meaningful boolean/decision versions, i.e. $g : \{0,1\}^* \rightarrow \{0,1\}$.

Eg. + has the decision version: (+, i) which asks for the i-th bit in the sum (a+b).

- A decision problem f has an associated *language* $L_f := \{x \in \{0,1\}^* \mid f(x) = 1\}$.

  eg. $L_{(+,i)} = \{(a,b) \in \{0,1\}^* \mid$ i-th bit in $(a+b)$ is $1\}$.

- The problems "computing a boolean f" & "deciding a language L_f" are the same!

Formalizing Machines

- A *Turing machine* (TM) M is described by a tuple (Γ, Q, δ) & two tapes

  finite *control* ... infinite *memory*

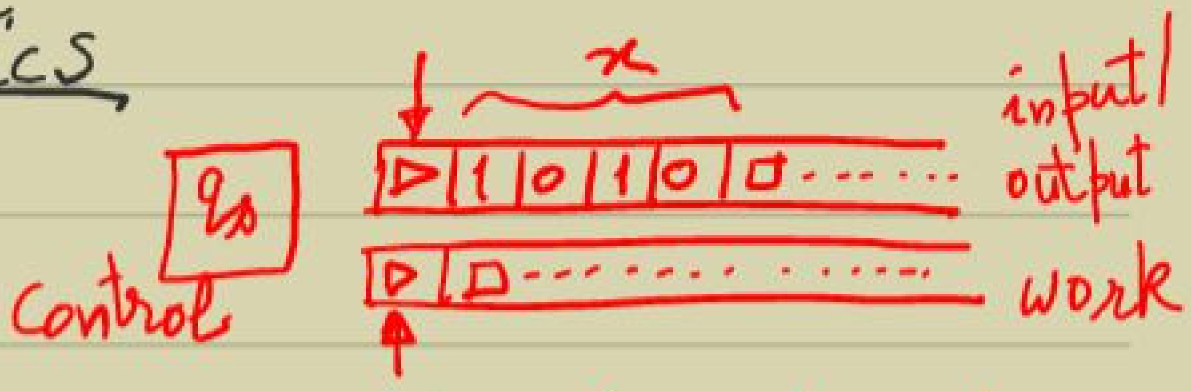
  Where,

  - Γ is the *alphabet*: It contains $\triangleright$ (start), $\square$ (blank) & other symbols (eg. 0,1).
  - Q is the set of *states*: It contains the states M can "be in". Q has at least q_s & q_f.

- δ is the transition fn.:

$$\delta: Q \times \Gamma^2 \longrightarrow Q \times \Gamma^2 \times \{S,L,R\}^2 .$$

TM semantics



(Start configuration)

- There are two tapes: input/output + work.

- Respectively, two heads.

- Say, the machine is at state $= q$, input cell value $= i$ & work cell value $= w$. Then, $\delta(q,i,w) = (q', i', w', \varepsilon_1, \varepsilon_2)$ means: In the next step (or configuration), state $= q'$, input cell value $= i'$, work $= w'$, input head moves via ε_1 (Stay, Left or Right), work head " " ε_2.

(No head moves to the left of ▷.)

- The computation "starts" with the start-configuration $(q_s, \triangleright, \triangleright)$ & "stops" (or halts) at the state q_f.

- The number of steps in the middle is the <u>time taken by M on x</u> to halt.

We disallow any writing on the first tape, except the output.

- The number of work-tape cells used is the <u>space taken by M on x</u>.

- Eg. <u>A TM to compute parity(x)</u>.

  We want $x_1 \oplus x_2 \oplus \cdots$. Roughly, the TM has steps:

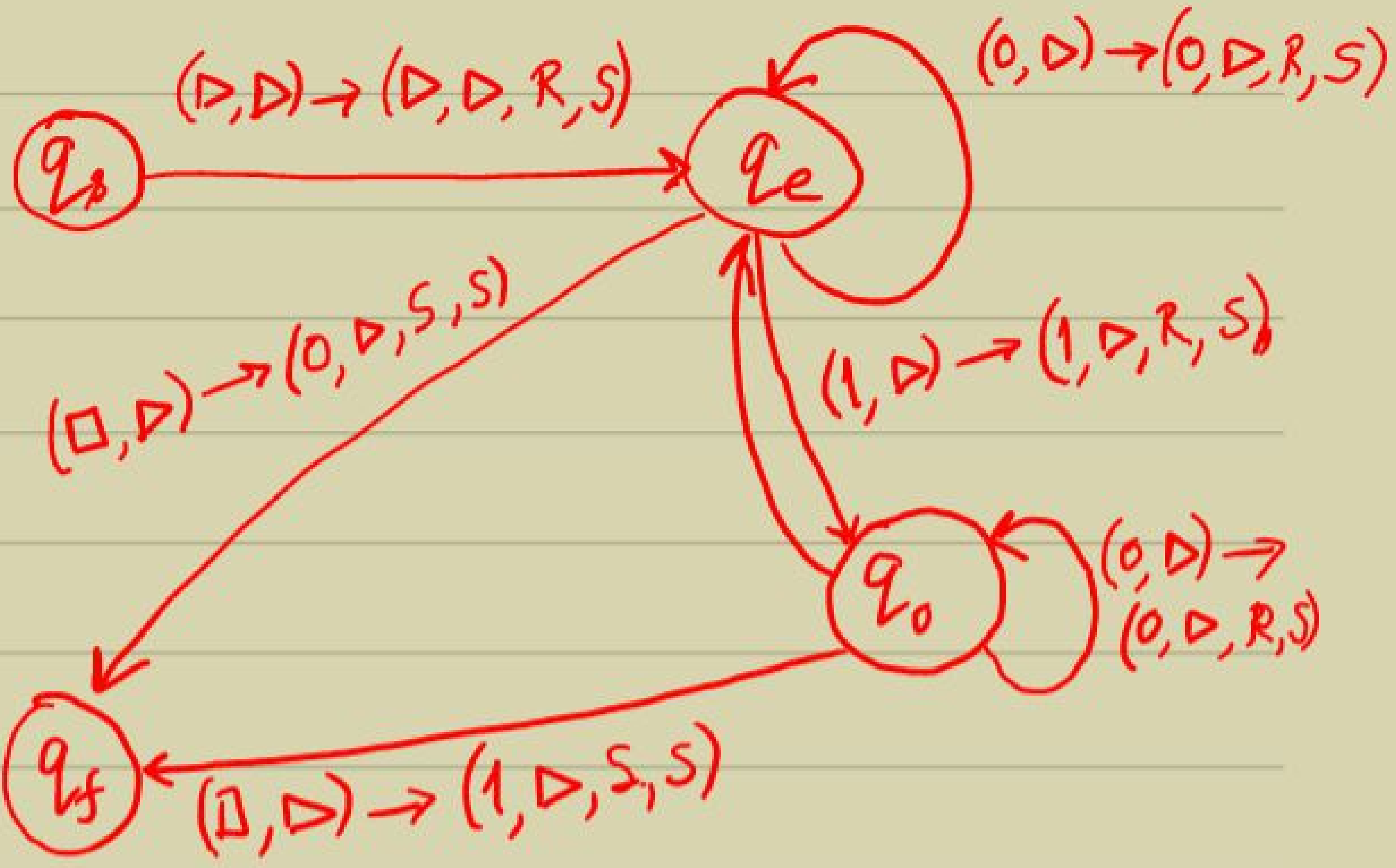
  (1) Read a bit of x.

  (2) If it's 1 flip the state.

  (3) continue till input-cell has □.

  (4) write 0 or 1, as the output, based on the state.

M:



- This is called a state-diagram of the TM M.

It describes δ, with the nodes being the states $Q = \{q_s, q_f, q_0, q_e\}$, & the edges specify a transition.

In this eg. the output bit is at the output-head in the end of the computation!

- The size of the state-diagram ≈ size of a C-program simulating this algorithm!

Time complexity (of a TM)

Definition: Let $f: \{0,1\}^* \to \{0,1\}^*$ & M be a TM computing f. Let $T: \mathbb{N} \to \mathbb{N}$ be a fn.
We say that "M computes f in T(n) time" if $\forall x \in \{0,1\}^*$, M on input x halts after $\leq T(|x|)$ steps & outputs $f(x)$.

- Similarly, define M computes f in S(n) space.

Asymptotic notation

- Let f, g be fns. from $\mathbb{N}$ to $\mathbb{R}_{\geq 0}$.

- **Big-Oh**: $f = O(g)$, if $\exists$ constants c, n_0 s.t. $\forall n \geq n_0$, $f(n) \leq c \cdot g(n)$.
- **Big-Omega**: $f = \Omega(g)$, if $g = O(f)$.
- **Theta**: $f = \Theta(g)$, if $f = O(g)$ & $f = \Omega(g)$.
- **small-oh**: $f = o(g)$, if $\lim_{n\to\infty} f(n)/g(n) = 0$.
- **small-omega**: $f = \omega(g)$, if $g = o(f)$.