CS 640 Course Presentation

Undirected Connectivity in Log Space - O. Reingold

Keerti Choudhary

Contents

Background

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- L $\subseteq$ SL $\subseteq$ NL.
- USTCON (Undirected s-t connectivity) was proved to be complete for class SL.
- This paper proves that USTCON can be solved in log space, thus shows L=SL.

Main Idea

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- Then s-t connectivity can be solved in $O(\rho \log D)$ space.
- So, we will convert G to a new graph satisfying
  - degree bounded by some constant c
  - diameter is $O(\log n)$.

Expander Graphs

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- A D-regular graph is ε-expander if any set S of size less than n/2 has closed neighbourhood of size at least $(1+\varepsilon)|S|$.
- Diameter of such graphs is bounded by $2\log_{1+\varepsilon} n$.
- Idea is to convert G to an expander graph whose degree is also bounded.

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- For every D, λ (λ<1), there exist ε>0 such that all D-regular graphs with second largest eigen value bounded by λ are ε-expander.

- To convert G to an expander graph we need to bound its λ .

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- As $\lambda(G) < 1 - 1/DN^2$, so squaring the graph $O(\log N)$ times would bound λ .
- But the cost is that degree will also be squared.

Bounding the degree

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Two methods -

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1. Replacement Product

Bounding the degree

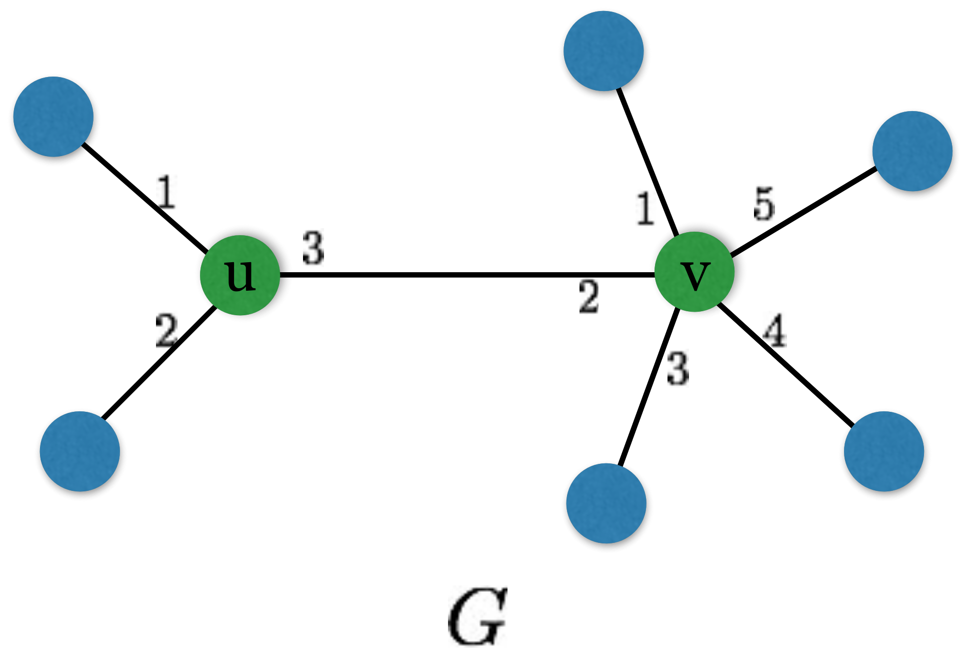
Two methods -

1. Replacement Product
2. Zig-zag Product

Rotation Map

In graph G, $Rot_G(u, i) = (v, j)$ iff,

- i^{th} vertex in adjacency list of u is v, and
- j^{th} vertex in adjacency list of v is u.



$$Rot_G(u, 3) = (v, 2)$$

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- Given a D-regular graph G, and a d-regular (d<<D) connected graph H on D vertices. Replace each vertex of G by H.

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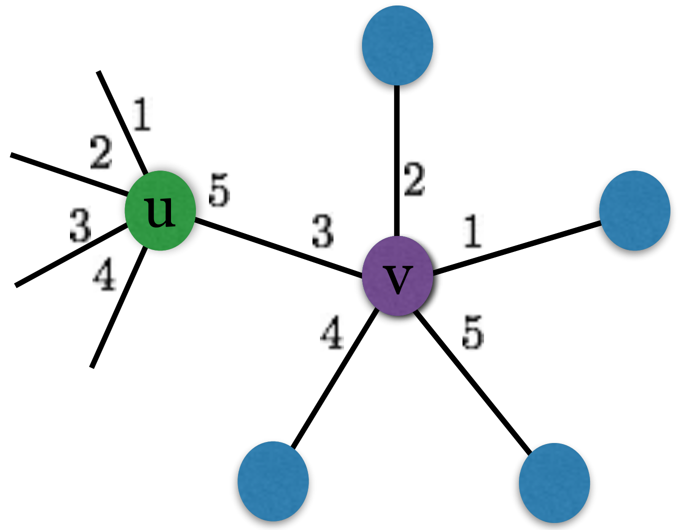

G

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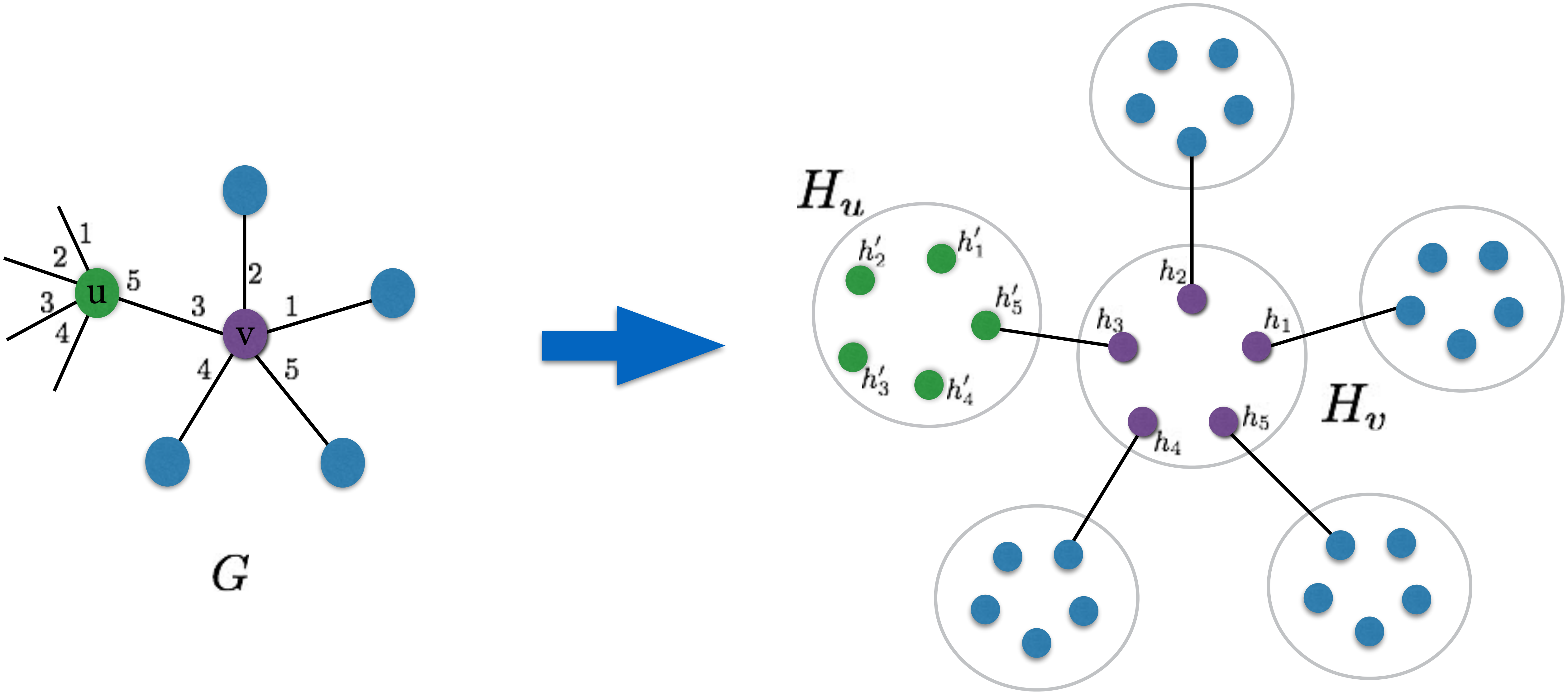

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Replacement Product

- Given a D-regular graph G, and a d-regular (d<<D) connected graph H on D vertices. Replace each vertex of G by H.

- New graph is a d+1 regular.

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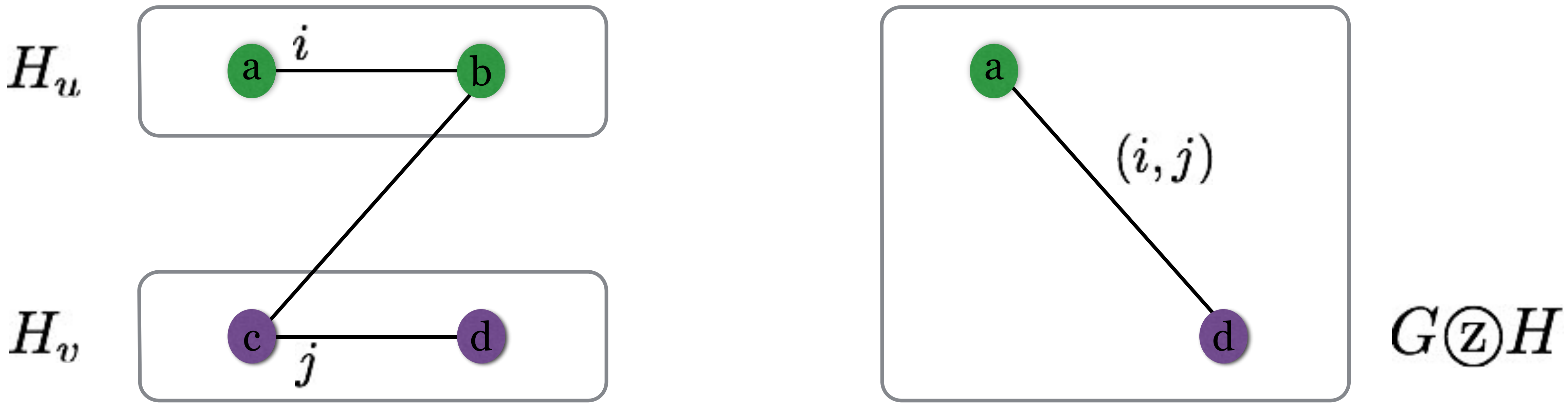

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- THEOREM - If G is an (N, D, λ)-graph and H is a (D, d, α)-graph, then

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- Here λ of the new graph is bounded above.

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- Proof Idea :
  - $\lambda(G_0) < 1 - 1/D^{16}N^2$
  - $\lambda(G_i) < max\{\lambda(G_{i-1})^2, 1/2\}$

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H is of constant size and can be encoded in TM.

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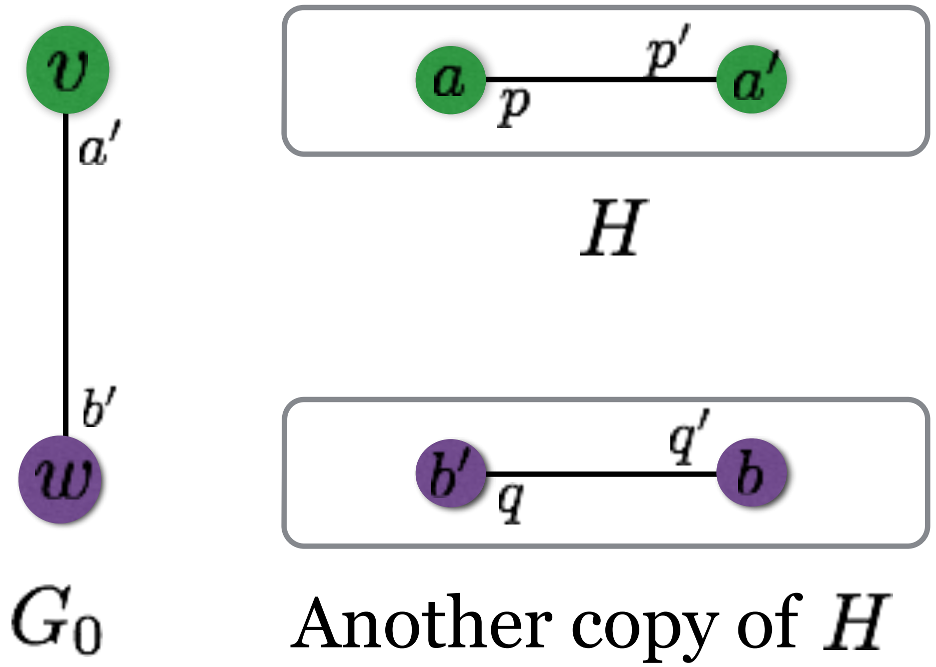

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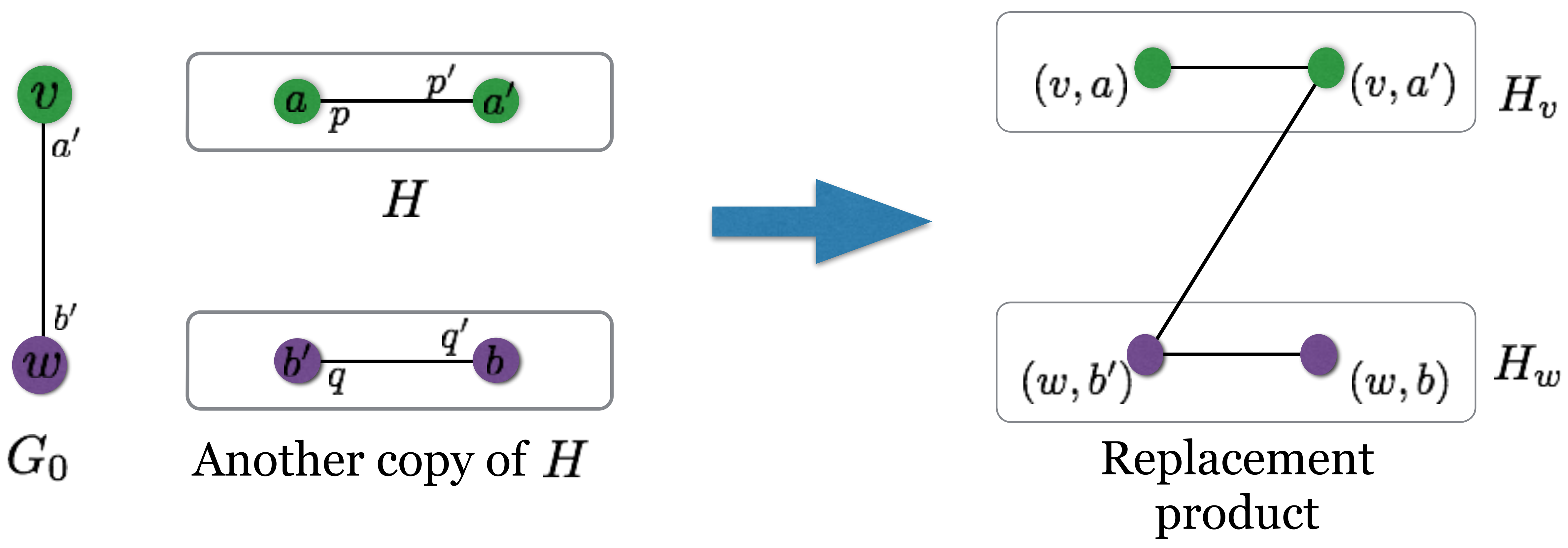
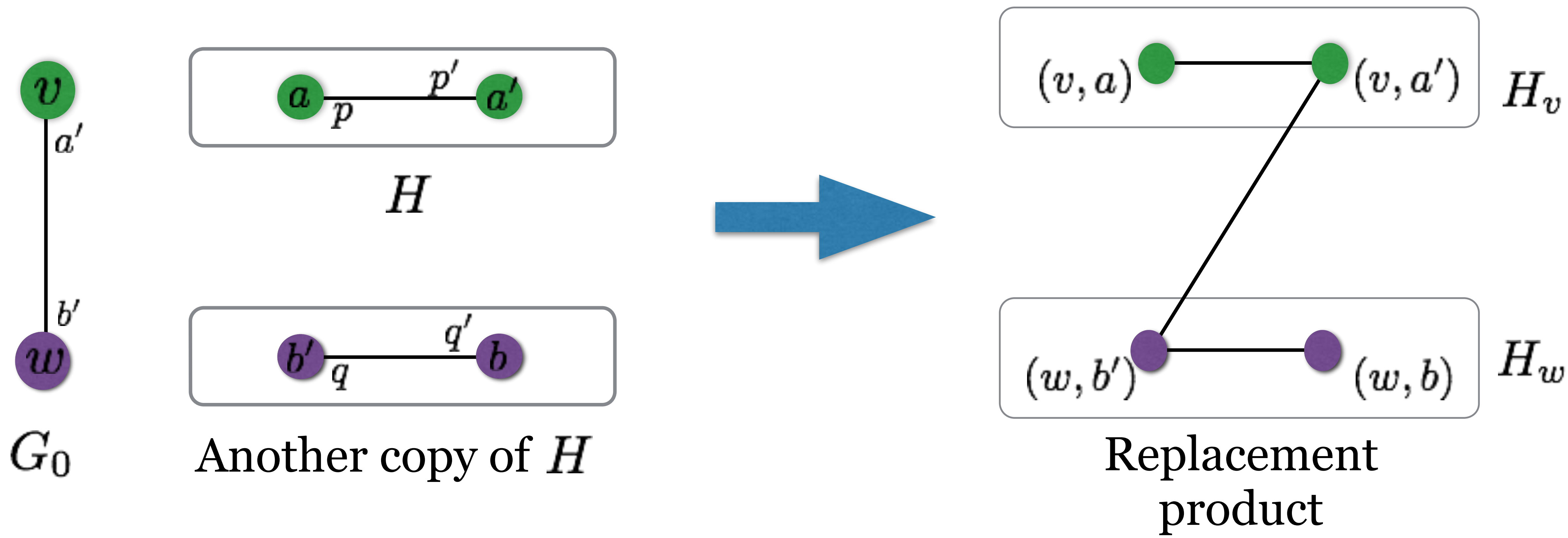
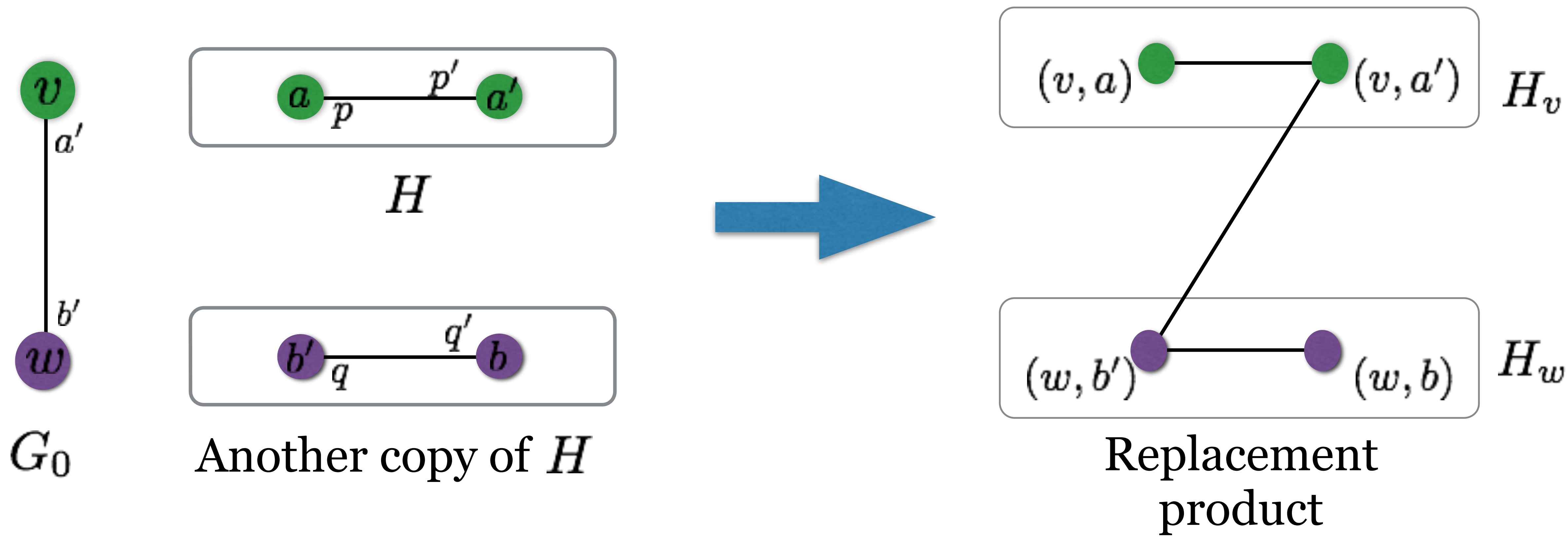




Now, $(v, a) \in [N] \times D^{16}$ and, we want to compute its $(p, q)^{th}$ neighbour



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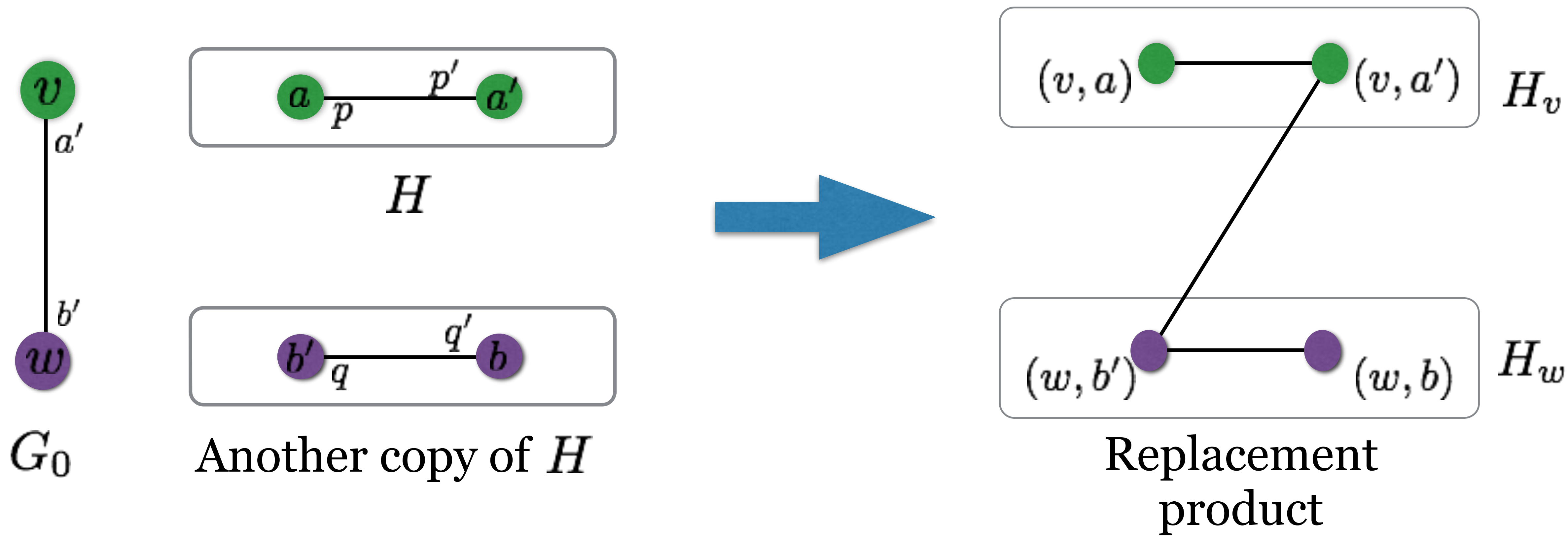

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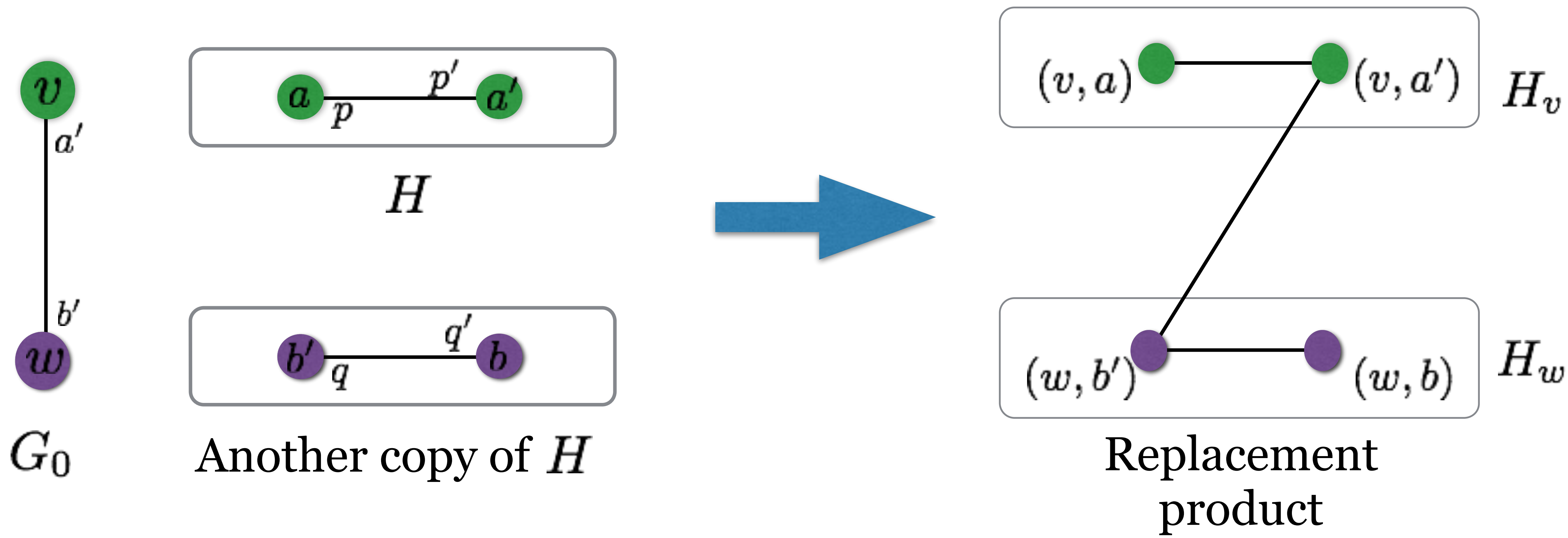

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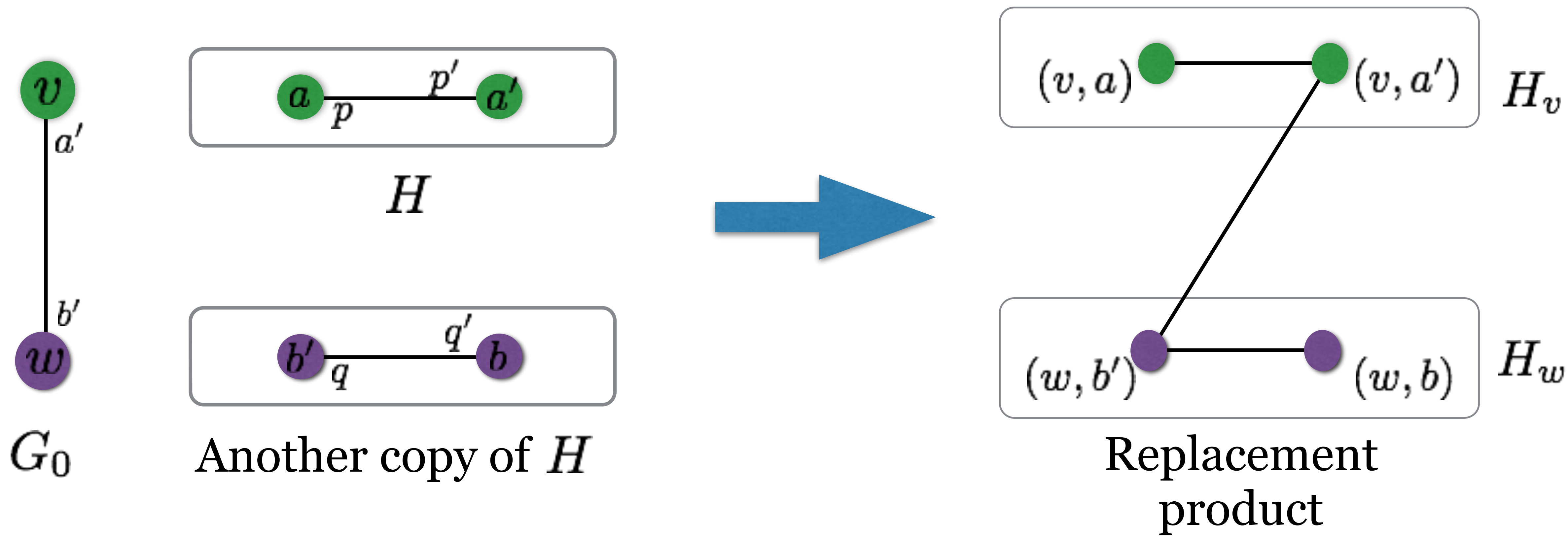

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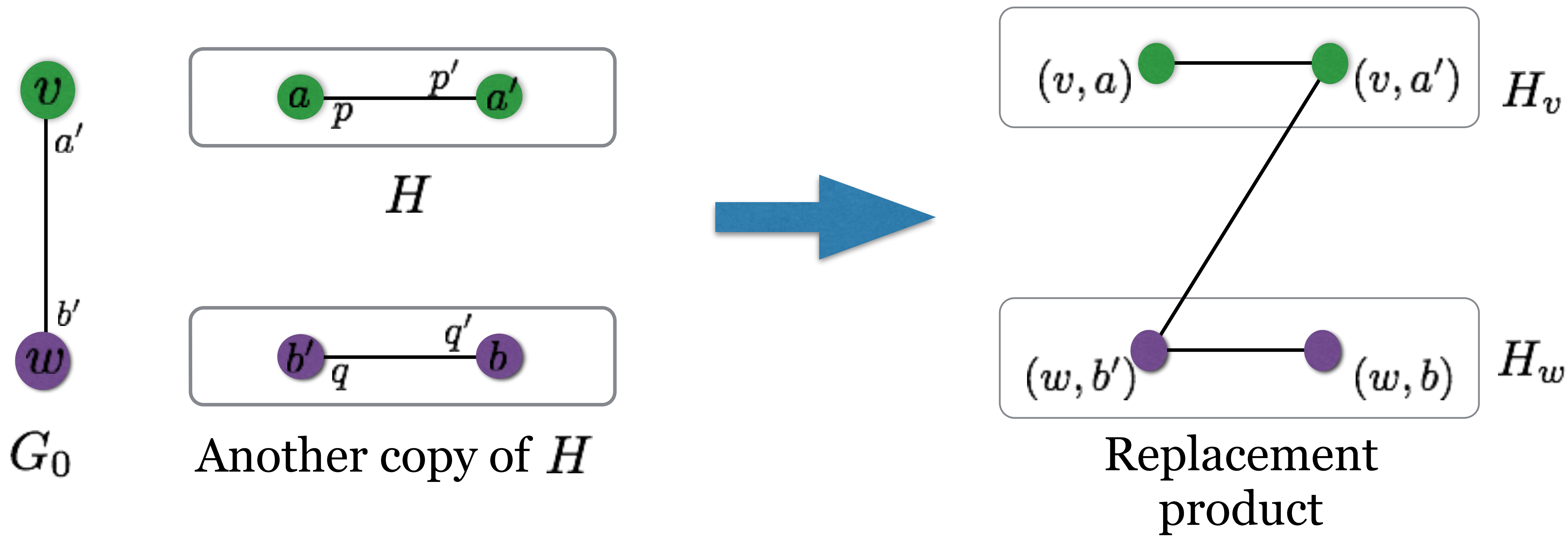

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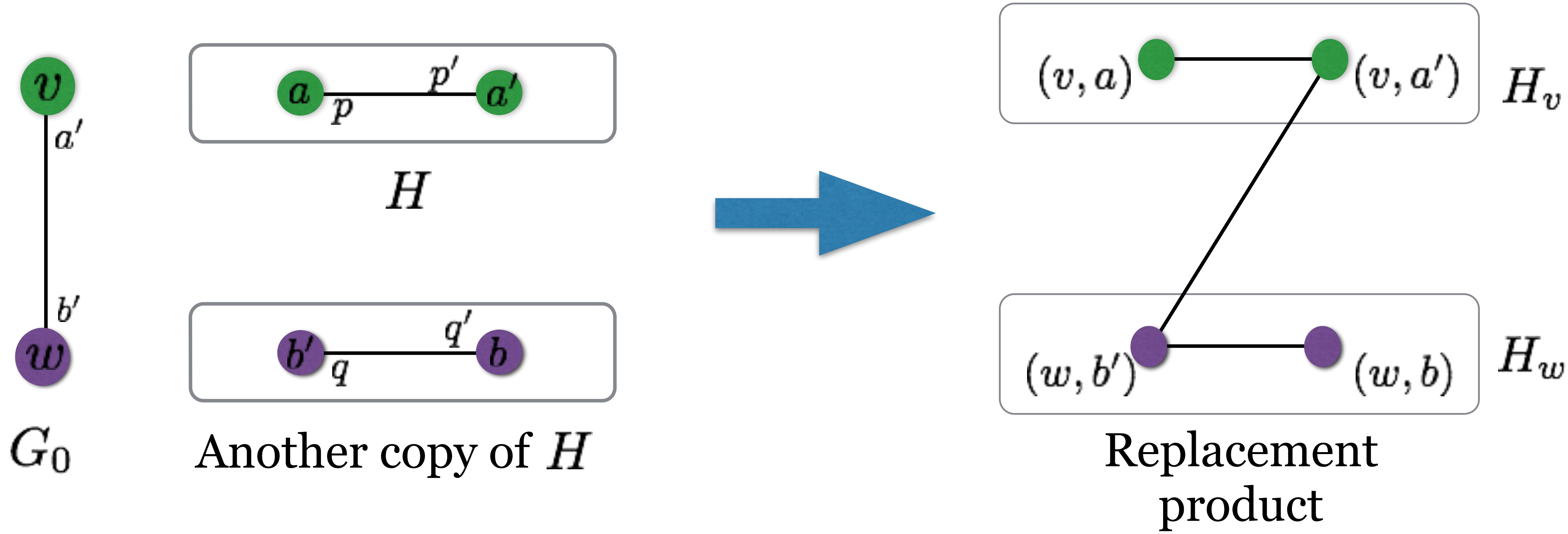

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- We need to compute the rotation map in log space.

Implementation

Main Algorithm

$Rot_{G_i}(v, a_0, ..., a_i)$:

1. $(p_1, q_1, p_2, ..., p_8, q_8) \leftarrow a_i$
2. For j=1 to 8
   - $a_{i-1}, p_j \leftarrow Rot_H(a_{i-1}, p_j)$
   - $v, a_0, ..., a_{i-1} \leftarrow Rot_{G_{i-1}}((v, a_0, ..., a_{i-2}), a_{i-1})$
   - $a_{i-1}, q_j \leftarrow Rot_H(a_{i-1}, q_j)$
3. $a_i \leftarrow (q_8, p_8, ..., p_2, q_1, p_1)$

THANK YOU