# Fibonacci Heap

- This is an advanced data structure. It is used when we want all operations (except deletion) to be very fast, eg. $O(1)$ time!

- For eg. in Dijkstra's algorithm we do $n$ Extract-min & $m$ ($\gg n$) Decrease-key operations.
  Trees take $O(m \lg n)$ time while Fibonacci heaps will take only $O(m + n\lg n)$.

- The heap is a collection of (rooted) trees with relaxed conditions; but each node stores a lot of pointers!

• Each node $x$ contains a pointer $p[x]$ to its parent & to a single $child[x]$.

- All the children of $x$ are linked in a circular doubly linked list (cdll):

  Each child $y$ (of $x$) has pointers left[y] & right[y]. If $y$ is the only child then these pointers equal $y$.

- Number of children is stored in deg[x].
- A special flag mark[x] indicates whether $x$ lost a child since $x$ was made a child of p[x]. (False)

  Newly created nodes have mark = F.
- A Fibonacci heap $H$ is accessed by the pointer min[H] that points to the root of a tree with min. key.
- The number of nodes in $H$ is n[H].
- The roots are also in a cdll.
- Key of $x \leq$ Key of any descendant of $x$.

- Let $t[H] :=$ #trees, $m[H] :=$ #marked nodes & $d(H) :=$ max deg of a node in $H$.

- Amortized analysis of heap operations:

Define $\phi(H) := t[H] + 2 \cdot m[H]$.

When a sequence of operations change the heap as $\phi =: H_0 \to H_1 \to \cdots \to H_\ell$ with $c_i$ being the actual cost in the $i$-th step, we define the amortized cost as

$$\hat{c}_i := c_i + \phi(H_i) - \phi(H_{i-1}).$$

(This may be negative for some $i$.)

▷ Total amortized cost $= \sum_{i=1}^{\ell} \hat{c}_i$

$$= \left(\sum_i c_i\right) + \phi(H_\ell) - \phi(H_0).$$

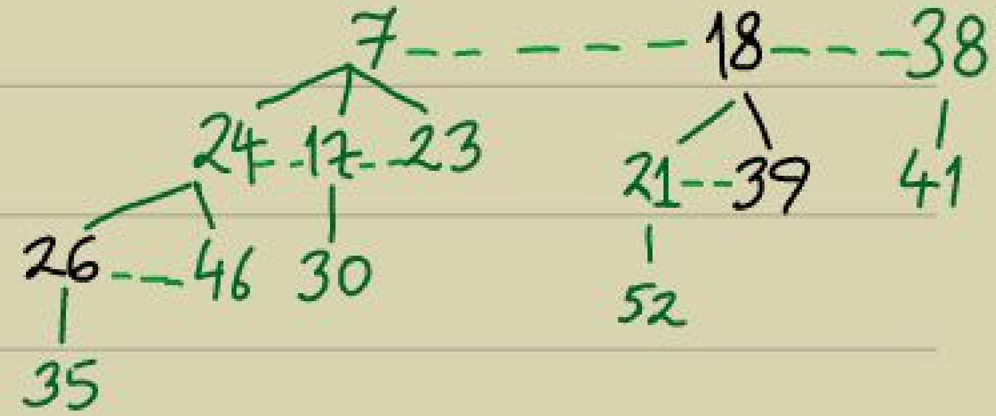
$$\geq \sum_i c_i = \text{actual cost.} \quad \left[\because \phi(H_0) = 0 \ \& \ \phi(H_i) \geq 0\right]$$

▷ Thus, (upper) bounding the amortized cost also bounds the actual cost.

Idea: There is not much structure in $H$ after the insert & decrease-key operations. This keeps them $O(1)$ time.

However, after extract-min an expensive step is done to consolidate the heap — in each row of siblings the degrees are distinct.

The amortized cost in this case is $O(d(H))$.




**Lemma**: Assuming the above $d(H) = O(\lg n)$, where $n = n[H]$.

**Proof**:

- Let $x$ be a node in $H$ with degree $d[x] =: k$.
- Let $y_1, \dots, y_k$ be the children of $x$ in the order they were linked to $x$.
- For $i \in [k]$, since $\deg[y_i]$ are distinct, we can write (up to permutation) that

$$\deg[y_i] \geq (i-1).$$

- Let $s_k$ denote the min. possible size of the tree rooted at $x$ with $deg[x] = k$.

$\Rightarrow s_k \geq s_0 + s_1 + \dots + s_{k-1}$

- It can be shown by induction that $s_k \geq F_{k+2}$ [Fibonacci numbers $F_0 := 0$, $F_1 := 1$, $F_{i+2} = F_{i+1} + F_i$.]

[It can be shown that $F_{k+2} \geq \phi^k$, where $\phi := (1+\sqrt{5})/2$.]

$\Rightarrow n[H] \geq s_k \geq \phi^k$.

$\Rightarrow k = O(\lg n)$. $\square$

- Let us now sketch the operations of Insert, Decrease-Key & Extract-Min.

```
Insert (H, x) {
    Create a node with key x;
    Add it in the root list;
    Update min[H] if required;
}
```

▷ Amortized cost $= O(1) + (t+1+2m) - (\underset{t(H)}{t} + 2\underset{m(H)}{m}) = O(1)$.

- Suppose in H we want to decrease the key of $x$ to $k$. ($key[x] > k$)

Let $y$ = parent of $x$.

- If $key[y] > k$ then $x$ is cut & added to the root list. [$Cut(H, x, y)$.]

- This requires a special process on the ancestors, called Cascading-cut:
  - If $y$ is unmarked then mark it.

  Else cut $y$ $Cut(H, y, z = p[y])$

  & Cascade on $z$.

▷ Suppose Cascade-Cut is called $c$ times. Then the amortized cost of Decrease-key is: $O(c) + \phi(H') - \phi(H)$

$= O(c) + (t+c) + 2(m-(c-1)+1) - [t+2m]$

$= O(c) + c + 2(2-c) = O(c) - c$

By scaling up $\phi$ we make the above $O(1)$.

→ Cost gets reduced by the negative potential change!

- Finally, we describe the most complicated operation — Extract-Min (H).

  It is here that the structure of the Fibonacci heap is secured.

- The min[H] root is removed & its children are added to the root list.

- Next, Consolidate (H) is called to link the roots that have the same degree.

  At the end, each sibling (in a row) has a distinct degree.

```
Consolidate (H) {
    for i = 0 to d(n[H])
        A[i] ← null;
    for w in the root list of H {
        x ← w; d ← deg[x];
        while A[d] ≠ null {
            y ← A[d]; // x & y have equal degree
```

if key[x] > key[y] then swap x,y;
Make y a child of x; Increment deg[x];
Mark[y] ← False;
A[d] ← null; d ← d+1;
} //end while
A[d] ← x;
} //end for
Update min[H];
}

- Amortized cost of Extract-Min(H):

$d(H) = O(\lg n[H])$ → The roots are $d(H)+t(H)$ many to begin with. (& in the end they're $\leq d+1$ many)

as other operations preserve it.

Each while iteration links two of them. Thus, the actual cost is $O(d(H)+t(H))$.

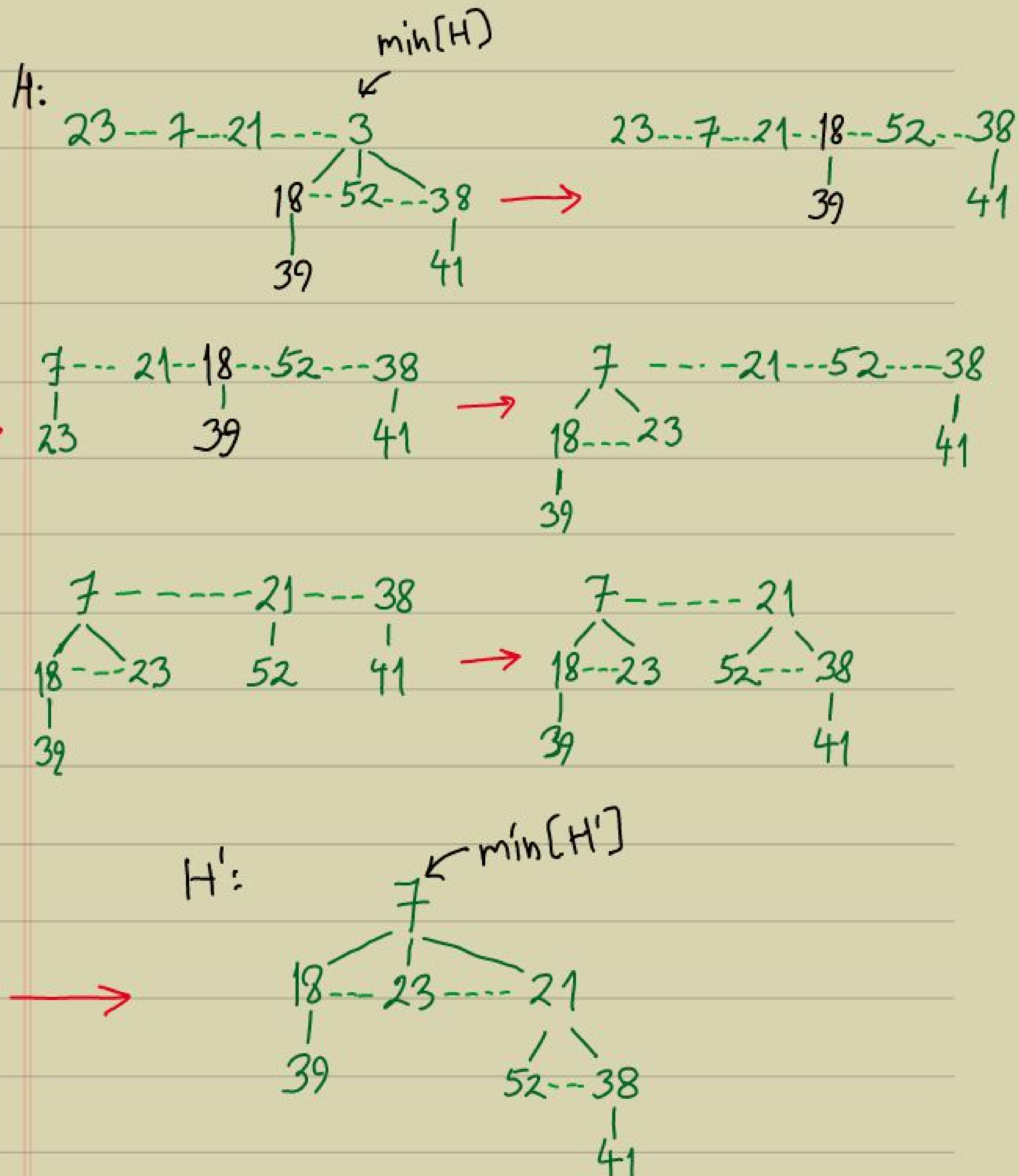
The amortized one is $O(d+t) + \phi(H') - \phi(H)$

$$= O(d(H)+t(H)) + [(d(H)+1) + 2m(H)] - [t(H) + 2m(H)] = O(d(H))$$

(By scaling up $\phi(H)$ function)

$$= O(\lg n[H]).$$

Eg.




<u>Exercise:</u> In each set of siblings (in the end), the degrees are <u>distinct</u>.