Stable Marriage Problem (SMP)

- There is a list of women $\{w_1, w_2, \ldots, w_n\}$ & men $\{m_1, \ldots, m_n\}$.

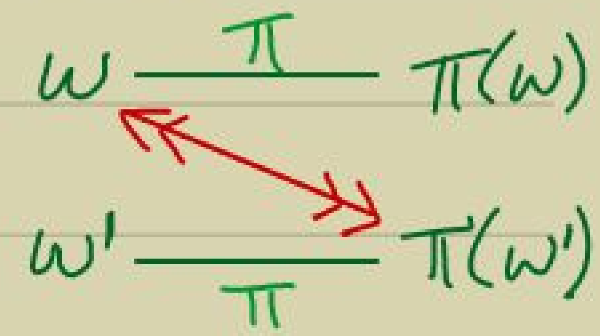
- Each woman (resp. man) has a ranking of preference of men (resp. women).

- We want to find a matching π such that the marriages are stable —

- Defn: Matching π is an unstable marriage if $\exists$ women w, w' & their men $\pi(w)$, $\pi(w')$ s.t.

  w prefers $\pi(w')$ over $\pi(w)$

  & $\pi(w')$ " w over w'.



- This problem has huge applicability in large scale: (Math./CS/Economics)
  1) Assignment of medical students to hospitals (eg. USA).
  2) JEE Counselling since 2016.
     (~20 x 110 "institutes" vs. 200,000 students)
  3) Assigning users to servers in a large distributed internet service.
     (10^5 servers vs. 10^9 users)

<u>Theorem</u>: [Gale-Shapley '62] It is <u>always</u> possible to find a stable marriage; doable in $O(n^2)$-time. [I.e. linear-time!]

- In 2012, Shapley & Roth got the Nobel prize for "market design".

- <u>Idea</u>: Man proposes, Woman disposes.
  (Or, deferred acceptance algorithm.)

- <u>Qn</u>: Who is happy?

- Single Man m proposes to a Woman w. w accepts m (temporarily) if she prefers him over her existing choice m'; in that case w rejects m'.

for each man

Input: Men M, Women W. Ordered lists L & P.

for each woman

Output: Perfect matching $M \to W$.

SMP:

```
S ← M ;
While (S ≠ φ) {
    m ← extract (S);
    w ← next(L(m));   // m prefers w
    m proposes to w;
    m' ← mate (w);   // m' may be null
    if ( w prefers m over m'; in P(w)) {
        w rejects m';  S ← S ∪ {m'};
        mate(w) ← m ;
        remove w from L(m)     ;
    }
    else {  "   "   " L(m) ; S ← S ∪ {m};}
} OUTPUT matching;
```

Qn: Does the algorithm terminate?

▷ Yes, in $O(n^2)$ iterations!

Pf:

• In each iteration, some $|L(m)|$ decrements.

$\Rightarrow$ In $O(n^2)$ iterations, $S = \phi$! $\square$

— Extra properties:

1) A man never proposes to the same woman twice.

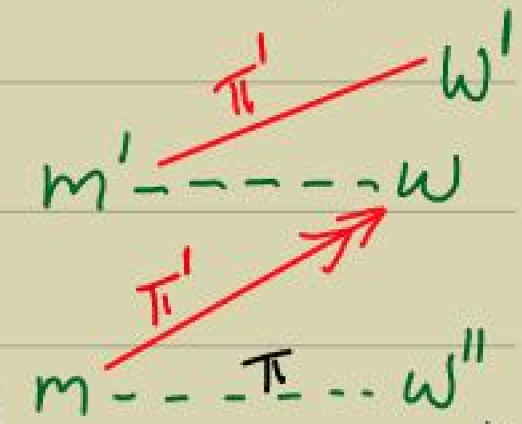
2) A woman once non-single, never becomes single.

3) A woman gets a better mate with each engagement.

4) Among all possible stable marriages, the output-matching is best for man m.

-Defn: valid(m) $:= \{w \mid \exists \pi, \pi(m)=w\}$ & best among them is best(m).

Thm: Algorithm is Men optimal, Women pessimal.



Proof:
- Suppose m is matched to w'' by π in SMP-algo.
- But, m prefers w over w''.
- Suppose $\exists$ stable marriage π' s.t. $\pi'(m) = w$ & $\pi'(m') =: w'$.
- When m proposed to w, she must have rejected (& preferred say m').

Call this event X. Let this be the first time when a valid partner rejected a man. (⇒ before this the best accepted all!)

By time X: (1) m' was rejected by every woman in $L(m')$ before w [$\because$ proposals]

(2) m' has not been rejected by w'.

[Note:- w' is a valid partner; invoke the defn of X.]

⇒ m' prefers w over w' [$\because$ (1) & (2)]

w " m' over m [$\because$ pseudocode]

⇒ π' is unstable, ↯ □