Eg. 3. Shortest paths with −ve wts.

- Recall that $G = (V, E, w)$ is a directed weighted graph with source vertex $s \in V$.
  We have to compute $\{\delta(s,v) \mid v \in V\}$ the set of shortest distances (& simple paths $P(s,v)$).

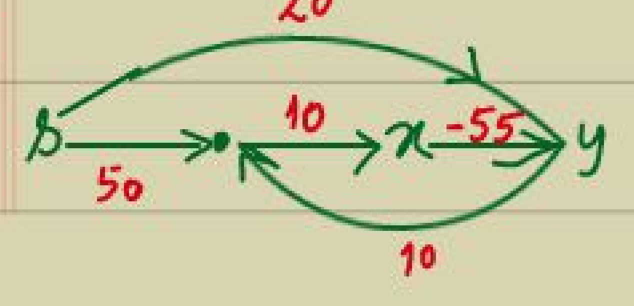
▷ Dijkstra solves the problem in $O(m + n \lg n)$ time, assuming non-negative weights.

- The non-negative wts. are critical in the analysis.
  What to do with negative wts.?

▷ A negative cycle means that distance can be as small as $-\infty$. (though the path is not simple)

- We will assume that there is no −ve cycle.

Bad eg.



▷ Subpath of $P(s,y)$ is not optimal!
$\because \delta(s,y) = 5$ while $\delta(s,x) = 40$.

Theorem: If $G=(V,E,w)$ has no negative cycle then shortest paths possess optimal substructure property.

Proof:

- Suppose the path $P(s,y)$ violates the optimal subpath property.

  $\Rightarrow \exists x \in P(s,y)$ s.t. distance of s to x along $P(s,y)$ is $> \delta(s,x)$.

  [Let x be the last such vertex.]

- If $P(s,x)$ is disjoint with $x \leadsto_{P(s,y)} y$, then composing the two we get a path shorter than $P(s,y)$. ↯

- Let $x' \in x \leadsto y$ be common between $P(s,y)$ & $P(s,x)$.

- We have: $w(q_1)+w(q_2) \leq w(p_1)$ & $w(p_2)+w(q_2) \geq 0$ $\Bigg\} \Rightarrow w(p_1)+w(p_2) \geq w(q_1)$

  which contradicts x. $\square$

– This makes the problem amenable to a recursive formulation.

The idea of Bellman & Ford:

- Consider paths $s \leadsto v$ with $\le i$ edges & define $P(v,i)$ to be a shortest one. Define $L(v,i)$ to be its length.

▷ Since we only consider simple paths, $s \leadsto v$ will have $< n$ edges.

- Let us attempt a recursive formulation:

<u>Case 1</u>: $P(v,i)$ has $< i$ edges:

$$L(v,i) = L(v,i-1).$$

$x \to v$

<u>Case 2</u>: $P(v,i)$ has i edges:

$$L(v,i) = \min_{(x,v)\in E} L(x,i-1) + w(x,v).$$

$$\Rightarrow L(v,i) = \min\Big(L(v,i-1), \min_{(x,v)\in E} L(x,i-1) + w(x,v) \Big).$$

<u>Base case</u>: $L(s,1) = 0$ & other $L(v,1) = \begin{cases} w(s,v), & \text{if } (s,v)\in E \\ \infty, & \text{else.} \end{cases}$

```
Bellman-Ford(s, G=(V,E,w)) {
    For each v ∈ V \ {s}
        if (s,v) ∈ E  L(v,1) ← w(s,v);
        else  L(v,1) ← ∞;
    L(s,1) ← 0;
    For i = 2 to n-1
        For each v ∈ V {
            L(v,i) ← L(v,i-1);
            For (x,v) ∈ E
                L(v,i) ← min(L(v,i),
                            L(x,i-1) + w(x,v));
        }
}
```

Exercise: Modify it to find $P(v,i)$.

Theorem (Bellman, Ford 1958): Given $(s, G=(V,E,w))$ without negative cycles, we can compute $\{P(s,v) \mid v \in V\}$ in $O(mn)$ time.

Proof:

- Dynamic programming implementation fills an $n \times n$ matrix $P := ((P(v,i)))$.

- The matrix updation shall take time $O(n) \cdot \sum_{v} \deg v = O(nm)$.

- Correctness is left as an exercise.

- Finally, $\{L(v, n-1) \mid v \in V\}$ is the output. $\square$

Exercise: It can be done in $O(n^2)$ space.

Exercise: Can you detect negative cycles?

▷ Shortest paths in the presence of negative cycles is NP-hard.

[Hint: Simple reduction from Ham-path.]

Eg.4. All-pairs shortest paths (faster)

- Dijkstra takes: $O(m+n\lg n)$ for single-source
  $\Rightarrow O(mn+n^2\lg n)$ for all-pairs.

- Bellman-Ford takes: $O(mn)$ for single-source
  $\Rightarrow O(mn^2)$ for all-pairs.

- Can it be solved faster?
  Negative weights allowed but no negative cycle.

How to exploit substructure?

- Floyd-Warshall's idea: To find a shortest path between vertices i & j, consider the max-index vertex k in such a path.

Defn: $P_k(i,j)$ is a shortest path $i \rightsquigarrow j$ with the max-vertex k in between.
$D_k(i,j)$ is its length.

- The recursive formulation is easy.
- <u>Case 1:</u> $P_k(i,j)$ indeed passes k.

$$\underbrace{i \to \bullet \rightsquigarrow \bullet \to k}_{P_{k-1}(i,k)} \underbrace{\to \bullet \rightsquigarrow \bullet \to j}_{P_{k-1}(k,j)}$$

$$\Rightarrow D_k(i,j) = D_{k-1}(i,k) + D_{k-1}(k,j).$$

- <u>Case 2:</u> $P_k(i,j)$ does not pass k.

$$\Rightarrow D_k(i,j) = D_{k-1}(i,j).$$

▷ $D_k(i,j) = \min\big(D_{k-1}(i,j),\ D_{k-1}(i,k) + D_{k-1}(k,j)\big)$, for $i,j \in [n]$ & $k \in [0 \ldots n]$.

▷ Base case: $D_0(i,j) = \begin{cases} w(i,j) & \text{, if } (i,j) \in E \\ 0 & \text{, } i=j \\ \infty & \text{, else} \end{cases}$

Adjacency matrix essentially

- What is the "matrix" in the dynamic programming implementation?

For each k use an $n \times n$ matrix.

- Qn: Can we manage with <u>one</u> matrix?

- There is an amazing reason:

Observation: 1) The k-th row/column of matrix D_{k-1} does not change, i.e.

$$D_k(i,k) = D_{k-1}(i,k) \ \& $$
$$D_k(k,j) = D_{k-1}(k,j).$$

2) For $i,j \in [n] \setminus \{k\}$, the (i,j)-th entry of matrix D_k changes only based on its k-th row/column.

$\Rightarrow$ We could store D_{k-1} & D_k in the same matrix.

- This yields a simple pseudocode:

```
FloydWarshall(G=(V,E,w)) {
    For i = 1 to n
        For j = 1 to n
            if ((i,j)∈E)  D[i,j] ← w(i,j);
            else if (i=j)  D[i,j] ← 0;
            else D[i,j] ← ∞;
    For k = 1 to n
        For i = 1 to n
            For j = 1 to n
```

..... (contd)

.... (contd.)

```
        if (D[i,j] > D[i,k] + D[k,j])
            D[i,j] = D[i,k] + D[k,j];
}
```

subscripts dropped

Lemma: At the end of <u>k-th iteration</u>, $D[i,j]$ is the length of shortest path using intermediate vertices $[k]$ (i.e. $D = D_k$)

Pf: (Exercise) $\square$

Exercise: Retrieve the shortest paths.

Theorem (Floyd, Warshall, 1962): Given a graph $G = (V, E, w)$, the all-pairs shortest paths are computable in $O(|V|^3)$ time.
$O(n^2)$-space

- Bellman-Ford iterates on the hops from the source s.
  Floyd-Warshall iterates on the max-index of an intermediate vertex.