Depth-first search - DFS traversal

- Keep the following applications in mind:
  1) Are all paths unique in a given G?
  2) Is G acyclic?

  [Tarjan'72] 3) Find the strongly connected components in G? I.e. a maximal $S \subseteq V$ s.t. $\forall i,j \in S$, $i \rightsquigarrow j$ & $j \rightsquigarrow i$.

  [Edmonds, Karp'72] 4) Maximum flow?
  5) Matching?

(Hopcroft & Tarjan'73) - DFS idea: Recursively explore a neighbor before moving to the next neighbor.

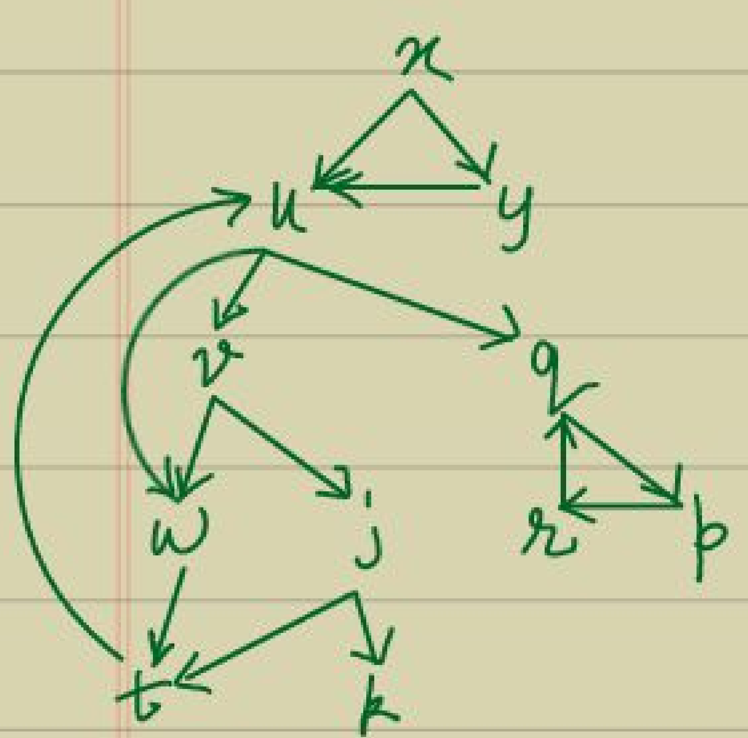
- Pseudocode:

```
DFS-graph (G=(V,E)) {
    For each v∈V visit[v] ← false;
    count ← 1;
    For each v∈V
        if (visit[v] = false) DFS(v);
}
```

```
DFS(v) {
                visit[v] ← true;
begin time →    D[v] ← count++;
                For each (v, w) ∈ E with visit[w] = false
                                                do DFS(w);
end time →      F[v] ← count++;
}
```

▷ $O(n+m)$ time complexity is easy to show.

– Example:



timestamp:

$D[u] = 1$
$D[v] = 2$
$D[w] = 3$
$D[t] = 4$
$F[t] = 5$
$F[w] = 6$

$D[j] = 7$
$D[k] = 8$
$F[k] = 9$
$F[j] = 10$
$F[v] = 11$
$D[q] = 12$
⋮

▷ Note the general relationship:

$D[u] < D[v] < F[v] < F[u]$

(Interval enclosed)
for tree edge (u, v)

- **Lemma:** (u,v) is a forward edge or an edge in the DFS-tree iff $D(u) < D(v) < F(v) < F(u)$ & $(u,v) \in E$.

Pf: • ($\Rightarrow$): is clear from the recursive nature of the DFS traversal.

• ($\Leftarrow$): In the DFS algorithm either v is explored as a neighbor of u; or, it is explored as a neighbor of w; in the latter case (u,v) is called a forward edge.

$\square$

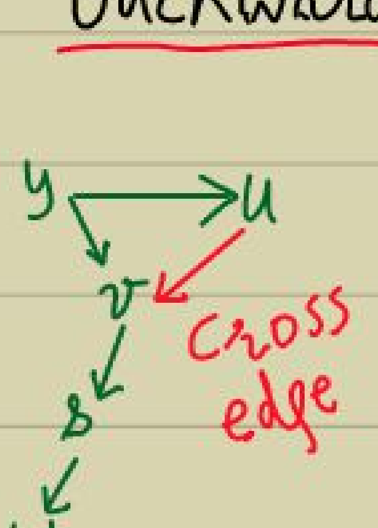
Lemma: The other kinds of non-tree edges are:

(Parenthesis theorem)

- cross edge (u,v): $(D[u], F[u])$ & $(D[v], F[v])$ are disjoint.

- backward edge (u,v): $D[v] < D[u] < F[u] < F[v]$.

Egs.



Application 1: Unique-path testing

- Input: Directed graph $G = (\overset{n}{V}, \overset{m}{E})$.

  Output: Yes iff $\forall u, v \in V$, $\exists$ at most one simple path from u to v.

- Easy algorithm:

```
For each u ∈ V {
    DFS(u);
    if (DFS-tree has a forward
    or a cross edge)
    then output non-unique-path;
}
output unique-path;
```

▷ Backward edges in DFS(u) do not yield multiple simple paths from u to any v.

▷ Time complexity is $n \times O(m+n)$.

Bringing it down to $O(n^2)$.

- Note that if $\exists u, v \in V$ s.t. in DFS(u):
  there are two backward edges from v
  then G is non-unique-path!
  Thus,

▷ G is unique-path $\Rightarrow \forall u \in V$, DFS(u) has no cross edge, no forward edge & $\forall v \in V$, at most one backward edge from v.

- This can be used to speed up the code:

```
DFS(u) {
        visit[u] ← true;
        D[u] ← count++;  //D[·], F[·] are initialized to -1.
        BackEdge[u] ← 0;
        For each (u,v) ∈ E {
            if (visit[v] = false)  DFS(v);
```

......(contd.)

$u \downarrow v$

:: forward/cross edge

```
else if (F[v]>0) output non-unique-path;
    else {
        BackEdge[u] ++;
        if(BackEdge[u] ≥ 2)      :: Two backward edges
            output non-unique-path;
    } //end else
} //end for
F[u] ← count ++;
} //end DFS(u)
```

Lemma: The algo., for DFS(u) above, takes $O(n)$ time.

Pf:

- The edges that are traversed are either DFS(u) tree edges, or ≤ 1 forward edge, or ≤ 1 cross edge, or ≤ 2 backward edges.

- This gives us a count of $n-1+1+n+1 = O(n)$ edges actually traversed. $\square$

Theorem: Doing this for all u, we get an $O(n^2)$ time algo to decide unique-path.

Application 2: Topological ordering in dag

- Can a backward edge appear in DFS-Graph(G), if G is a dag?
  - No; else we've a cycle.

Theorem: Testing whether G is a dag is doable in $O(n+m)$ time.

- Let $G \overset{(V,E)}{=}$ be a dag & $u \in V$.
  Let v, w be vertices in DFS(u).
  - If (v,w) is a DFS-tree edge then $F[v] > F[w]$.
  - If (v,w) is a forward edge then $F[v] > F[w]$.
  - If (v,w) is a cross edge then $F[v] > F[w]$.

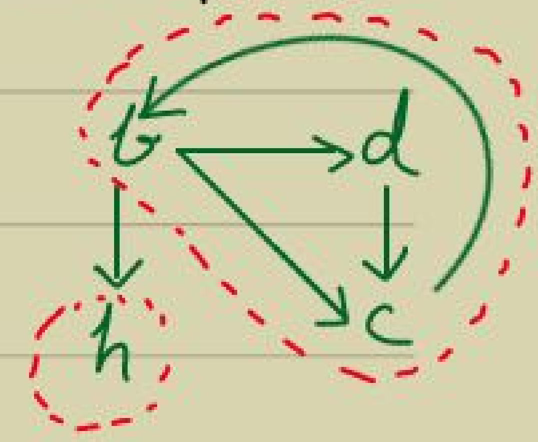
Theorem: Decreasing order of finish time $F[\cdot]$ gives us a topological ordering on a dag G in $O(n+m)$ time.

Application 3: Compute strongly connected comp.

- **Defn:** Let $G=(V,E)$ be a directed graph. Vertices $u, v \in V$ are strongly connected if $u \rightsquigarrow v$ & $v \rightsquigarrow u$.

  A maximal subset of vertices in V, with every pair strongly connected, is called a strongly connected component. (SCC)

- Qn: In an undirected graph?



▷ Using repeated DFS we can find the strongly connected components in $n \times O(m+n)$ time.

Pf: (Exercise) □

- Qn: Can this be improved?

- Yes, in $O(m+n)$ time.

  We'll discuss Kosaraju (1978)'s idea.

- Defn: The root of a scc S is the vertex which is visited first in DFS.

$\Rightarrow$ $\triangleright$ Root of a scc S is the vertex that finishes last (among S).

- This gives an idea to spot the root of some scc:

Do DFS on G, order V in the decreasing order of $F[\cdot]$ & pick the first vertex u.

$\Rightarrow$ u is the root of a scc in G, say S.

- Qn: How do we find S?

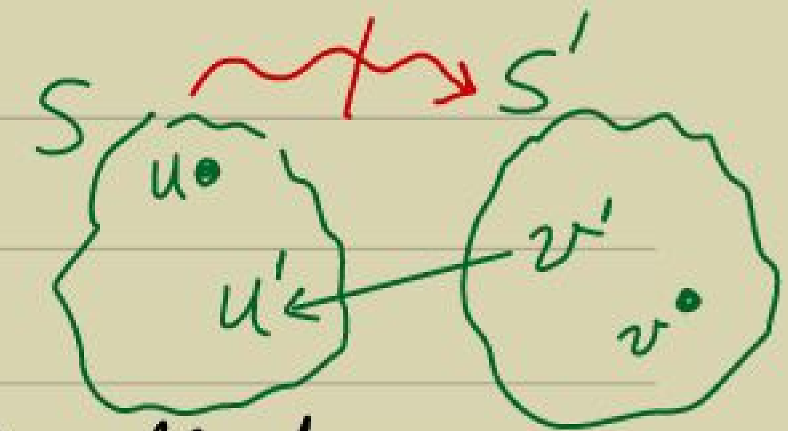
Let S' be another scc & $v' \in S'$.

Lemma: $\forall\, u' \in S,\ v' \in S',\ (v', u') \notin E$.

Pf:

- Let v be the root of S'. ..... (Contd.)

- Let (v', u') be an edge.



- Case 1: [DFS visits v' first] Clearly, u' will be a descendant of v' in the DFS tree.

  $\Rightarrow$ u will be a descendant of v $\Rightarrow F[u] < F[v]$, which is a contradiction!

- Case 2: [DFS visits u' first]

  Since v' is unreachable from S, first the exploration of u' will finish & then the DFS of S' can begin.

  $\Rightarrow F[u] < F[v]$.

$\Rightarrow$ We get a contradiction in all cases. $\square$

– Thus, edges could only go out of S.

...... (contd.)

..... (contd.)

- This gives us the idea to reverse the edges in G & consider $G^r := (V, E^r)$:

▷ S^r is a scc in G^r & there is no edge going out of S^r.

▷ DFS on u in G^r gives us S^r!

Final algorithm for SCC

- Do DFS on G & store V in the decreasing order of $F[\cdot]$.
- Compute G^r by reversing the edges.
- Initialize $\forall v \in V$, $\begin{cases} \text{SCC-found}[v] \leftarrow \text{false}; \\ \text{SCC-num}[v] \leftarrow -1; \end{cases}$
- $i \leftarrow 1$;
- For each $v \in V$ {

```
    if(scc-found[v] = false) {
        Do DFS(v) in G^r;
        Let the DFS-tree have vertices A;
        For each x ∈ A {
            scc-found[x] <- true;
            scc-num[x] <- i;
            remove x from G^r;
        }
        i++;
    }
}
```

Theorem: SCC is computable in $O(m+n)$ time.

Pf: (Exercise)