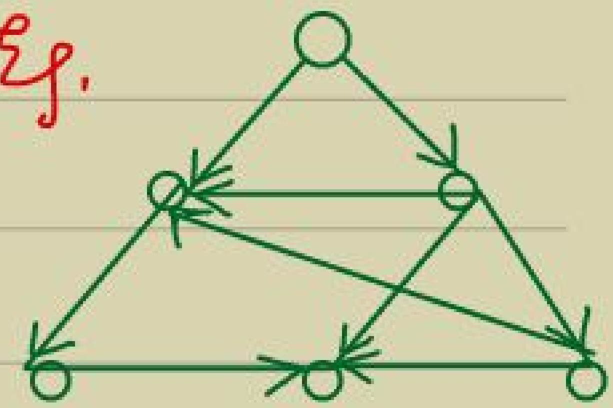
Directed Acyclic Graph (dag)

- **Defn:** A directed graph $G=(V,E)$ is acyclic if there is no cycle in it.

Eg.



τ:

- **Defn:** A topological ordering is a map $\tau : V \to [n]$ s.t. $\forall (u,v) \in E$, $\tau(u) < \tau(v)$.

- Questions: Does τ exist?
  Computable efficiently?
  Its use?

▷ τ is used to compute the shortest paths, & even count the number of paths, in a DAG in linear time!

Lemma: A dag has a vertex of in-deg $= 0$.

Pf:

- Let G be a dag.
- Start with any $v \in V$.

$u \longrightarrow v$
- If it has a pre-neighbor u, i.e. $(u, v) \in E$ then move to u.

  Else v has in-deg$=0$ & done.
- Continuing this way the process stops in $\leq |V|-1$ steps with an in-deg$=0$ vertex.
- Note: The process cannot get in a cycle. $\square$

Theorem: Dag has a top. ordering τ.

Pf:

- Let $v \in V$ be an in-deg$=0$ vertex.
- Define $\tau(v) := i := 1$.
- Remove v from V & its outgoing edges from E. Graph G remains a dag.
- Repeat the process with the least label $i \leftarrow i+1$.
- At $i = |V|$ all the vertices are labelled.

- Moreover, $\forall (u,v) \in E$, $\tau(u) < \tau(v)$ is maintained in every iteration, for the labelled vertices u, v. $\square$

▷ Naively, τ takes $O(|V|^2)$ time to compute.

– Can this be improved?

– Yes: Find the in-deg $=0$ vertex faster using a queue data structure. (array)

– Given dag $G = (V, E)$.

- Create a queue $Q \leftarrow \phi$;
- For each $x \in V$

elts. in Q have in-deg$=0$.

    if (in-deg$(x) = 0$) Enqueue(x, Q);

- $i \leftarrow 1$;
- While $(Q \neq \phi)$ {

    $v \leftarrow$ Dequeue(Q);

    $\tau(v) \leftarrow i$; $i \leftarrow i+1$;

(contd.)

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• For each (v, x) ∈ E {
      in-deg(x) ← in-deg(x) − 1;
      if (in-deg(x) = 0) Enqueue(x, Q);
  }
return τ;
```

- Correctness: It is labelling in-deg = 0 vertex & removing it.

- Time: The two loops have number of runs $\leq |V| + \sum_{v \in V} \deg(v) = |V| + 2|E|$.

<u>Theorem</u> (Knuth 1968): Topological sorting can be done in $O(|V| + |E|)$ time.

<u>Theorem</u>: Single-source shortest paths, in a dag, can be found in linear time.

<u>Proof</u>:

• Given a dag $G = (V, E, w, s)$ do topological sorting.

- Initialize $L(s) \leftarrow 0$ & $L(v) \leftarrow \infty, \forall v \in V$.
- $\forall u \in V$ in order & not behind s {

$s \rightsquigarrow u \longrightarrow v$

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    ∀(u,v) ∈ E {
        if (L(v) > L(u) + w(u,v))
            L(v) ← L(u) + w(u,v);
    }
}
```

- This works because, inductively, we are correctly computing $L(u)$ for nodes closer to s in the topological order.

- Time: $O\left(|V| + \sum_{u \in V} deg(u)\right) = O(|V| + |E|)$. □

Theorem: Number of paths $s \rightsquigarrow t$, in a dag, can be computed in linear time.

Proof:

- Given a dag $G = (V, E, s)$ do topological sorting.

- Initialize $L(v) \leftarrow 0$, $\forall v \in V$;
- $\forall u \in V$ in order & not behind s {

    $\forall (u,v) \in E$ {

        $L(v) \leftarrow L(v) + L(u)$;

    }

  }

- By induction on the #hops from s, we can see that $L(u) =$ #paths $s \rightsquigarrow u$.

- Time: $O(|V| + |E|)$. $\square$

<u>Theorem</u>: Given dag $G = (V, E)$ & vertices $x_1, \dots, x_k$ we can compute the number of paths of the form $x_1 \rightsquigarrow x_2 \rightsquigarrow \dots \rightsquigarrow x_{k-1} \rightsquigarrow x_k$ in linear time.

<u>Proof</u>:

- Do a topological sort on G.
- Check that $x_1 < x_2 < \dots < x_k$.
  Otherwise the answer is zero.
- Let n_i be the number of vertices between x_i to x_{i+1}, $\forall i \in [k-1]$.

- Let m_i be the respective number of edges.
- We can compute $p_i := \#\text{paths } x_i \rightsquigarrow x_{i+1}$ in $O(n_i + m_i)$ time.

$\Rightarrow \{p_i \mid i \in [k-1]\}$ can be computed in $\sum_{i \in [k-1]} O(n_i + m_i) = O(|V| + |E|)$ time.

$\Rightarrow$ Number of paths of the form $x_1 \rightsquigarrow x_2 \rightsquigarrow \dots \rightsquigarrow x_k$ is $\prod_{i \in [k-1]} p_i$.

$\square$