Partially Ordered Sets

- Relations over a given set.
  - Eg. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$: "$\leq$" : $a \leq b$ or $a \geq b$ or $a = b$.
    (or, $a < b$ or $a > b$)
- reflexive, symmetric, antisymmetric, transitive.
  → equivalence relations.
  → posets / totally ordered sets.

Defn: Relation R from S to T is a subset of $S \times T$; i.e. $R \subseteq S \times T$, or $R \in 2^{S \times T}$.

- $\underline{S \times T} := \{(a,b) \mid a \in S,\ b \in T\}.$

▷ Relation may <u>not</u> be a function/map.
▷ Function/map is a relation.

- eg. less/greater/equal (on numbers),
  divides ("),
  friends (on people),
  contained in (on sets).
- We'll be interested in relation of S to S.

- Defn: Relation R on set S to S is:
  - Reflexive: $\forall s \in S$, $(s,s) \in R$.
  - Symmetric: $\forall s,t \in S$, $(s,t) \in R \Rightarrow (t,s) \in R$.
  - Antisymmetric: " , (s,t) & $(t,s) \in R$ $\Rightarrow s = t$
  - Transitive: $\forall s,t,u \in S$; $(s,u),(u,t) \in R \Rightarrow (s,t) \in R$.

Exercise: Pick a subset of axioms. Construct an R that doesn't have the remaining properties?

Qn: Is a relation always symm. or antisymm.?

- Eg. $R = \{(1,2)\}$ ✗ ; $R = \{(1,2),(2,1),(1,3)\}$ ✓

Qn: What properties equality (=) satisfy?

↳ All four!

Equivalence Relations

- Defn: R is an equivalence if R is: reflexive, symm., transitive.

- "mod m" = R on $\mathbb{Z}$: $a \equiv b \pmod{m}$ if $a\%m = b\%m$

  i.e. $m \mid (a-b)$.

▷ "mod m" is an equivalence relation.

Theorem: Equiv. relation R, on S, partitions S into equivalence classes!

Pf: • Consider $R(s) := \{t \in S \mid (s,t) \in R\}$, for $s \in S$.

▷ $(s,t) \in R \iff R(s) = R(t)$.

Pf: ($\Leftarrow$): $R(s) = R(t) \Rightarrow t \in R(s) \Rightarrow (s,t) \in R$.

($\Rightarrow$): $(s,t) \in R$ & $u \in R(s) \Rightarrow (s,u), (u,s) \in R$

$\Rightarrow (u,t) \in R \Rightarrow u \in R(t)$

$\Rightarrow R(s) \subseteq R(t) \Rightarrow \dots \Rightarrow R(s) = R(t)$.

▷ $(s,t) \notin R \Rightarrow R(s) \cap R(t) = \phi$. □

Pf: • Assume $(s,t) \notin R$ & $u \in R(s) \cap R(t)$.

$\Rightarrow (s,u), (t,u) \in R \Rightarrow (s,t) \in R \Rightarrow \lightning$

$\Rightarrow R(s) \cap R(t) = \phi.$ $\square$

$\triangleright \bigcup_{s \in S} R(s) = S.$
$\Downarrow s$

$\Rightarrow \{\text{Distinct } R(s)\}$ partition S. $\square$

<u>Observation:</u> Any partition of S defines an equivalence relation on S.

Pf: • Say, $S =: S_1 \sqcup S_2 \sqcup \cdots \sqcup S_c$

• Define $(a,b) \in R$ if $\exists i \in [c]$, $a, b \in S_i$. $\square$

- Eg. Color the vertices of a square R/B.
  - Two colorings are related if second can be obtained from the first, using: rotation / reflect / rotate-reflect.
  - It's an <u>equivalence</u> relation.

Q: What are its classes?

eg. {RRRR}, {RRRB, BRRR, RBRR, RRBR},
{RRBB, ...}, {RBRB, BRBR, ...}, ...

Partial Order

- Now, we move to inequalities.

Defn: • Relation R is a partial order if it's: reflexive / antisymm. / transitive.

• Partial order R is total order if $\forall s,t \in S$: $(s,t) \in R$ or $(t,s) \in R$.

- eg. $(\mathbb{R}, \leq)$ is total order. Qn: $(\mathbb{R}, <)$ ✗
  $(\mathbb{R}, \geq)$ " " " .

- Eg. $(\mathbb{C}, \leq)$ not total, as $(0, \sqrt{-1})$ is incomparable.

$\hookrightarrow$ is partial order.

- Eg. $(2^{[n]}, \subseteq)$ not total, as $(\{1\}, \{2\})$ ....

$\hookrightarrow$ is partial order.

- Notation: • Partial order on S is denoted by $(S, \leq)$.

• "$a < b$" denotes $a \leq b$ & $a \neq b$ (in S).

▷ #(total orders on S) $= |S|!$

$|S| < \infty$.

- eg. $(\mathbb{C}, \leq)$: $\underline{a \leq b}$ if $|a| \leq |b|$

$\not\Rightarrow$ is <u>not</u> anti symm., eg. $(\sqrt{-1}, -\sqrt{-1})$

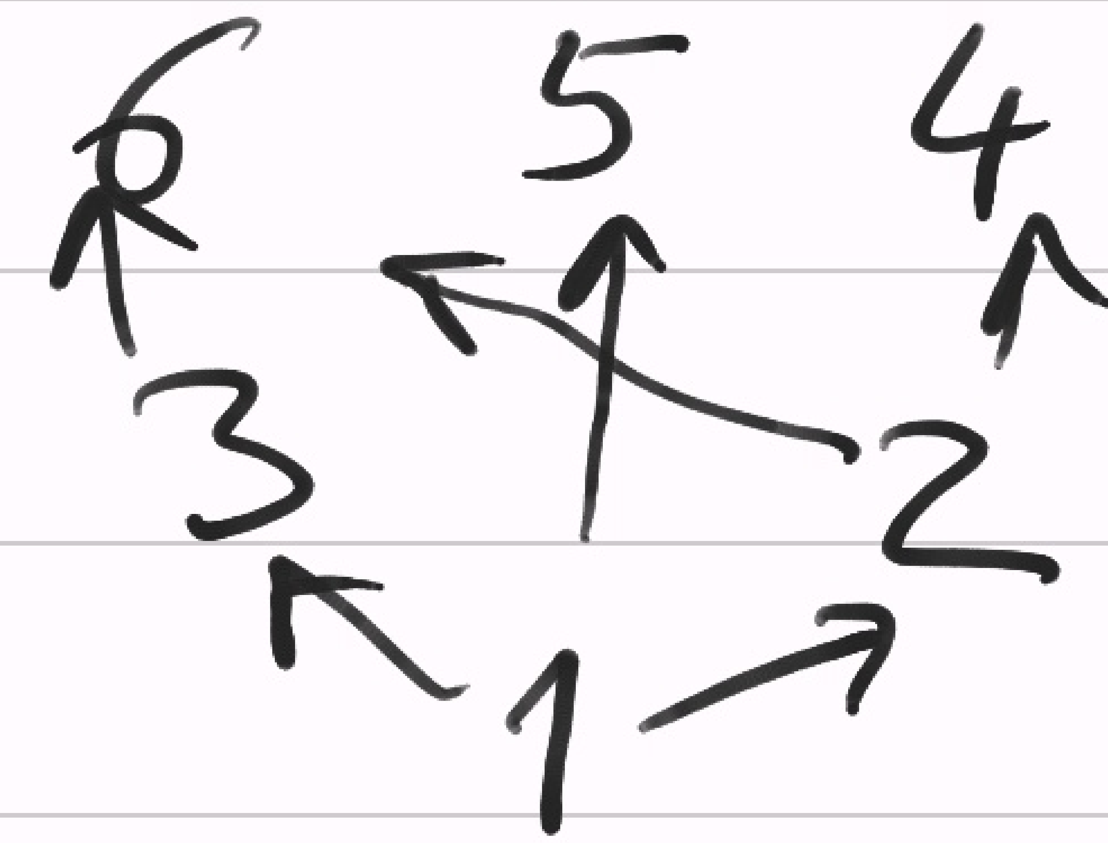
$\Rightarrow$ is <u>not</u> an order.

- <u>Qn</u>: Is there another partial order on $\mathbb{C}$?

- We can represent finite posets as a diagram:

Hasse Diagram (for poset)

- Let R be partial order on S.
- Connect $x, y \in S$ by a line if $x < y$ & $\nexists z \in S : x < z < y$. (skip self-loops)
- Eg. $(\mathbb{N}, \leq)$: $0 \to 1 \to 2 \to 3 \to \cdots$
- Eg. $(\mathbb{N}, \leq := \text{divides})$:



$\triangleright$ ($\forall x \in S$, if there is unique y s.t. "$x \to y$" appears) $\Longleftarrow$ R is total order. (Ex.)

→ Hasse diagram removes the edges which can be obtained from reflexivity & transitivity.

- **Defn:** • An element $x \in S$ maximal if no element appears on "top" of x.
  • An element $x \in S$ minimal if no element appears "below" x.

→ Are these unique? No.

Chains & Anti-chains

- Defn: • Chain $C \subseteq S$ s.t. any two elements in C are comparable.
  • Anti-chain $A \subseteq S$ s.t. no two elements in A are comparable.

- eg. What's the longest chain in $(2^{[n]}, \leq)$?
  ↳ eg. $C = \{\phi, \{1\}, \{1,2\}, \{1,2,3\}, \ldots [n]\}$.

Ex.: longest is $(n+1)$.

- Qn: Longest anti-chain ?

▷ All subsets of size $\lfloor n/2 \rfloor$ is anti-chain.
▷ " " " " $\lceil n/2 \rceil$ " "
$\Rightarrow$ both have length $\binom{n}{\lfloor n/2 \rfloor} \approx 2^n$.

<u>Theorem</u> (Sperner, 1928): Longest anti-chain is only one of these. [$\Rightarrow$ n even, it's <u>unique</u>.]

<u>Pf:</u> • Defn: A maximal chain (<u>m-chain</u>) C be $\phi = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_{n-1} \subset S_n = [n]$
▷ #(m-chains) $= n!$.
— Let <u>A</u> be an anti-chain.

▷ $|C \cap A| \leq 1$. [≤ 1 element of C in A.]

- Let's count #(m-chains containing $S \subseteq [n]$):
- Say, $C = \{ \phi = S_0 \subset \cdots \subset \underline{S} \subset \cdots \subset S_n = [n] \}$ size $=: k$

▷ #(m-chains with S) $= k! \cdot (n-k)!$ (Prod. Rule)

$\geq \lfloor \frac{n}{2} \rfloor ! \cdot \lceil \frac{n}{2} \rceil !$ (Ex.)

- Consider m-chains: $C_1, C_2, \ldots, C_{n!}$
- Our anti-chain $A =: \{ S_1, S_2, \ldots, S_a \}$

▷ $\forall i$, S_i appears in $\geq m := n! / \binom{n}{\lfloor n/2 \rfloor}$ chains.

$$\Rightarrow \underbrace{\#\,\text{green-parts}}_{a\,=:\,|A|} \leq n!/m = \binom{n}{\lfloor n/2 \rfloor}$$

$\triangleright$ $|A| \leq \binom{n}{\lfloor n/2 \rfloor}$.

- **Note:** The above proof shows that to get largest $|A|$, $\forall i$, $|S_i| = \underbrace{\lfloor n/2 \rfloor}_{k\,:=}$ or $\underbrace{\lceil n/2 \rceil}_{=:\,k'}$.

[Pf: Make each "$\leq$" to "$=$" in the above calculations.]

↳ Thm done for even n.

$\triangleright$ Odd $n \Rightarrow \forall i$, $|S_i| = k$ or $k+1 = k'$.

Lemma: If a longest A contains an S_i of size k, then it contains all k-subsets.

Pf: • Let S be a k-subset in our longest anti-chain A.

• Let $S' \neq S$ be some k-subset of $[n]$.

• $S =: S_0 \subset T_1 \supset S_1 \subset T_2 \supset \cdots \subset T_\ell \supset S_\ell =: S'$

Size $=$ k $\quad k+1 \quad k \quad k+1 \quad k+1 \quad k$

$\{1,2\}$

$\{1,2,4\}$

$\{1,4\}$

$\{1,4,3\}$

$\{3,4\}$

$\Rightarrow$ $T_1 \notin A$. Qn: Is $S_1 \in A$?

– Consider an m-chain $\cdots \supset T_1 \supset S_1 \supset \cdots$

$\because T_1 \notin A$, hence $S_1 \in A$.

– Repeat this: $S_2 \in A$, $S_3 \in A$, $\ldots$, $S_\ell \in A$.

$\Rightarrow$ Every k-subset is in A. $\square$

$\Rightarrow$ [n odd] $A = \binom{[n]}{k}$ or $\binom{[n]}{k+1}$.

set of all the k-subsets

set of all $(k+1)$-subsets

- **Key**: Relation between chains & anti-chains. $\square$

[Mirsky thm, '71] If the longest chain in poset has length r, the poset has a partition into r anti-chains.

[Dilworth's thm, '50] If longest anti-chain in poset has length r, then the poset has a partition into r chains.