- Qn: Finding minimum wt. spanning tree?
edges are weighted

Tours on a graph

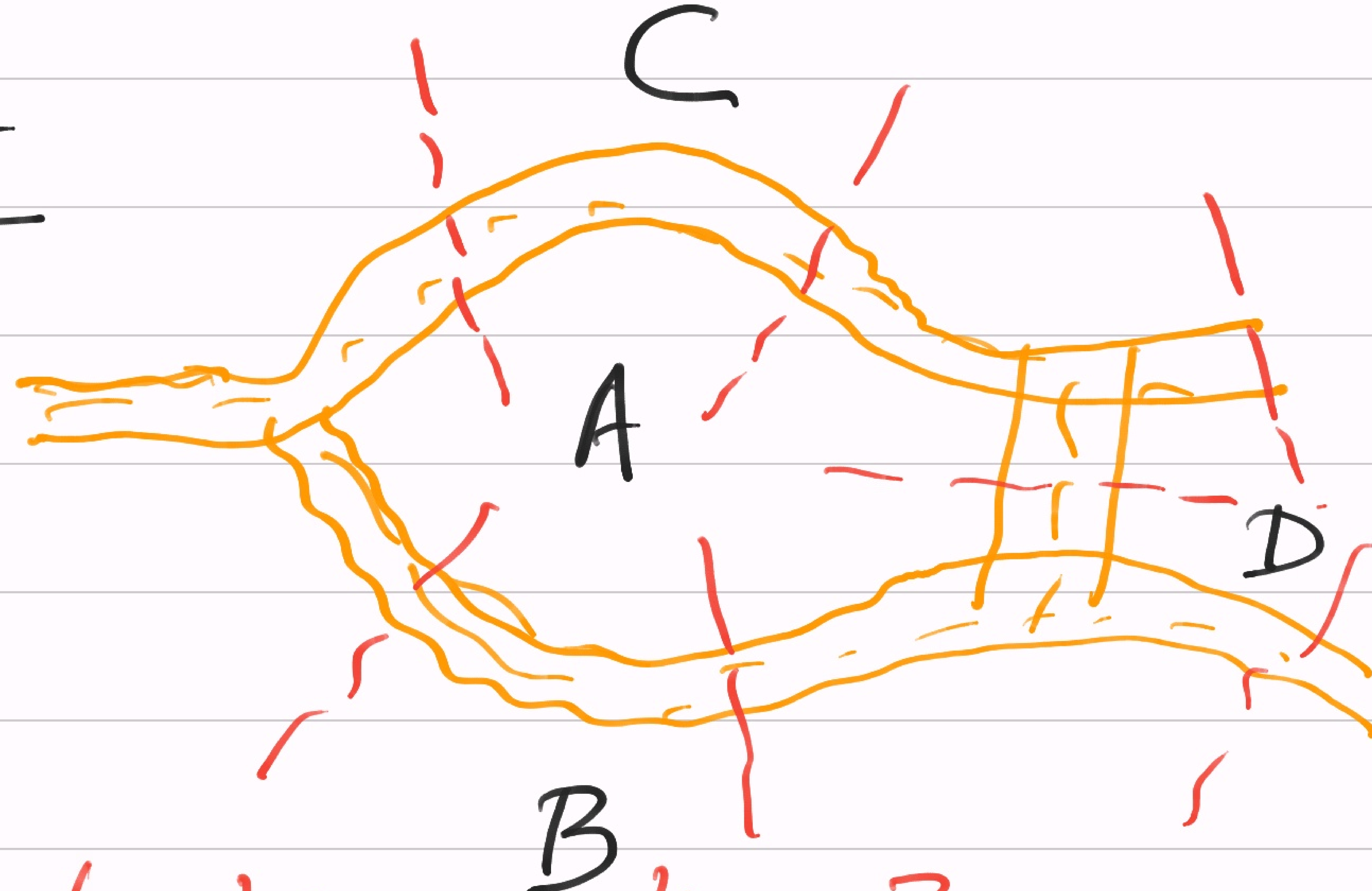
- Given a connected graph, is it possible to visit all vertices (resp. edges) without revisiting? (source = destination)

→ Sometimes it's easy, sometimes hard?

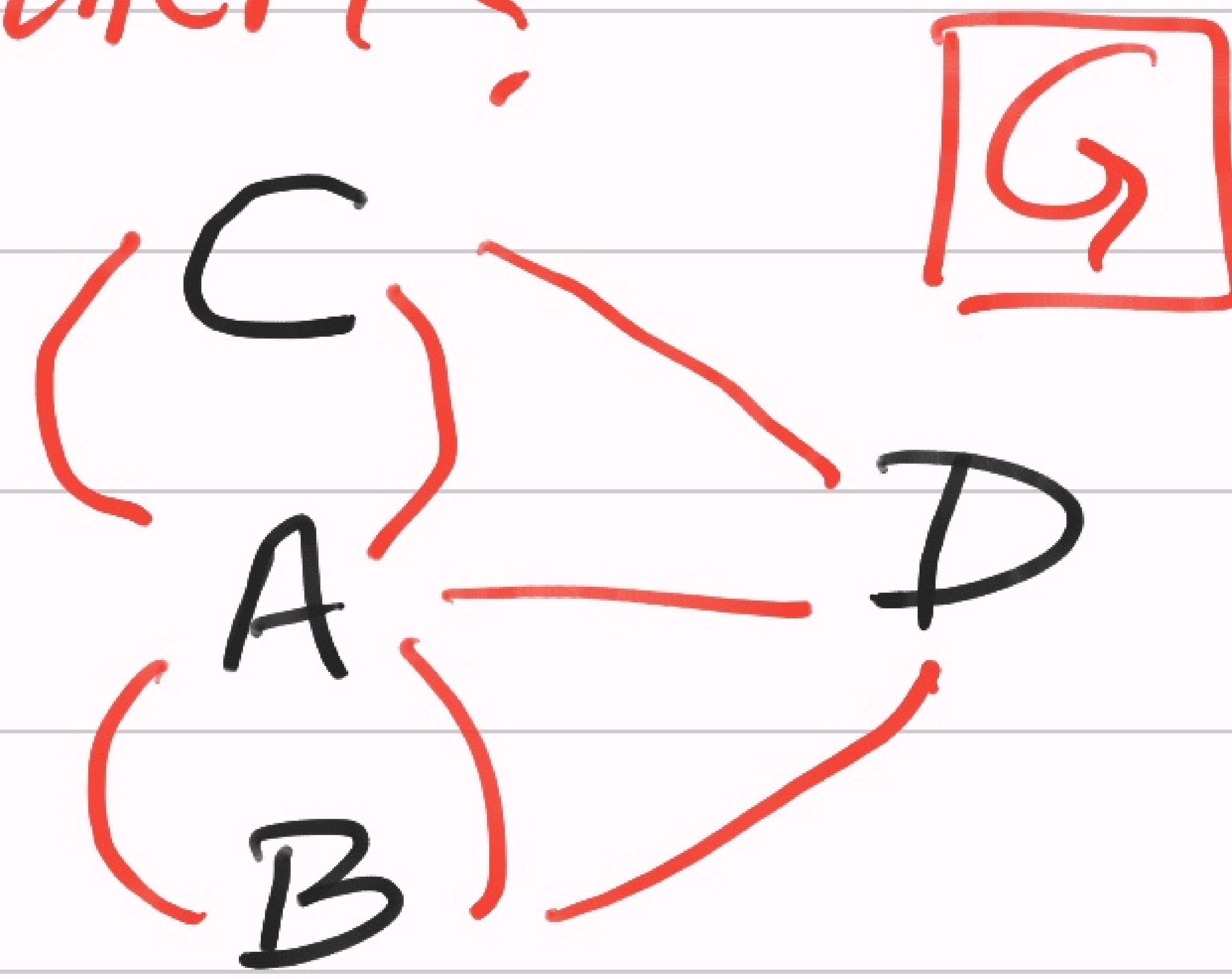
1) Touring vertices → Hamiltonian circuit.
2) " edges → Eulerian circuit.

Eulerian Circuit

(Königsberg bridges problem)



Qn: $\exists$? tour of the bridges, without repeating them?

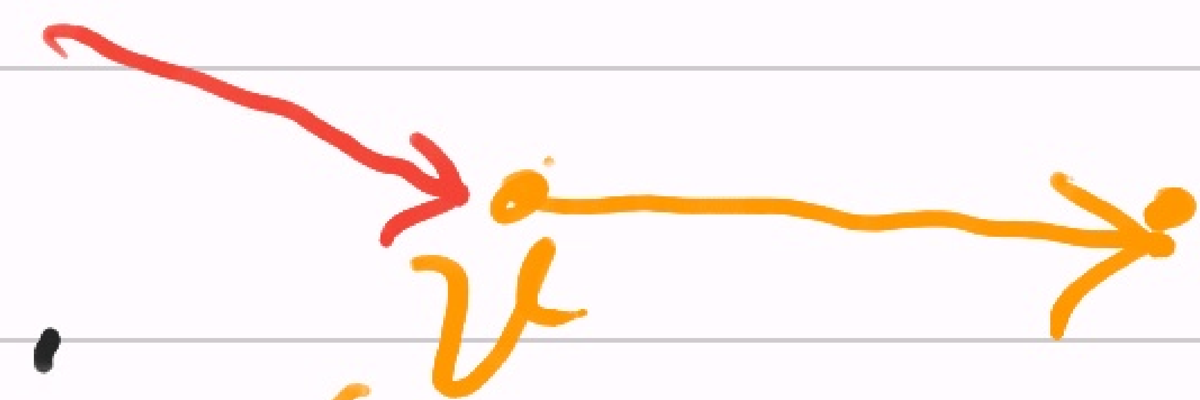


Defn: Eulerian circuit is a tour of G covering each edge uniquely.

<u>Theorem</u> (Euler, 1736): A connected graph has an Eulerian circuit iff all vertices have <u>even</u> degree.

<u>Pf</u>: ($\Rightarrow$): Since we don't reuse an edge in the tour, entry-exit-edges at v get paired (& different).

$\Rightarrow \deg(v) = \text{even}, \ \forall v \in V(G)$.



($\Leftarrow$): Let G be a graph with $\deg(v) = \text{even}$, $\forall v \in V(G)$. How to find the <u>tour</u>?

- Start with some vertex w and get a <u>closed</u> walk.

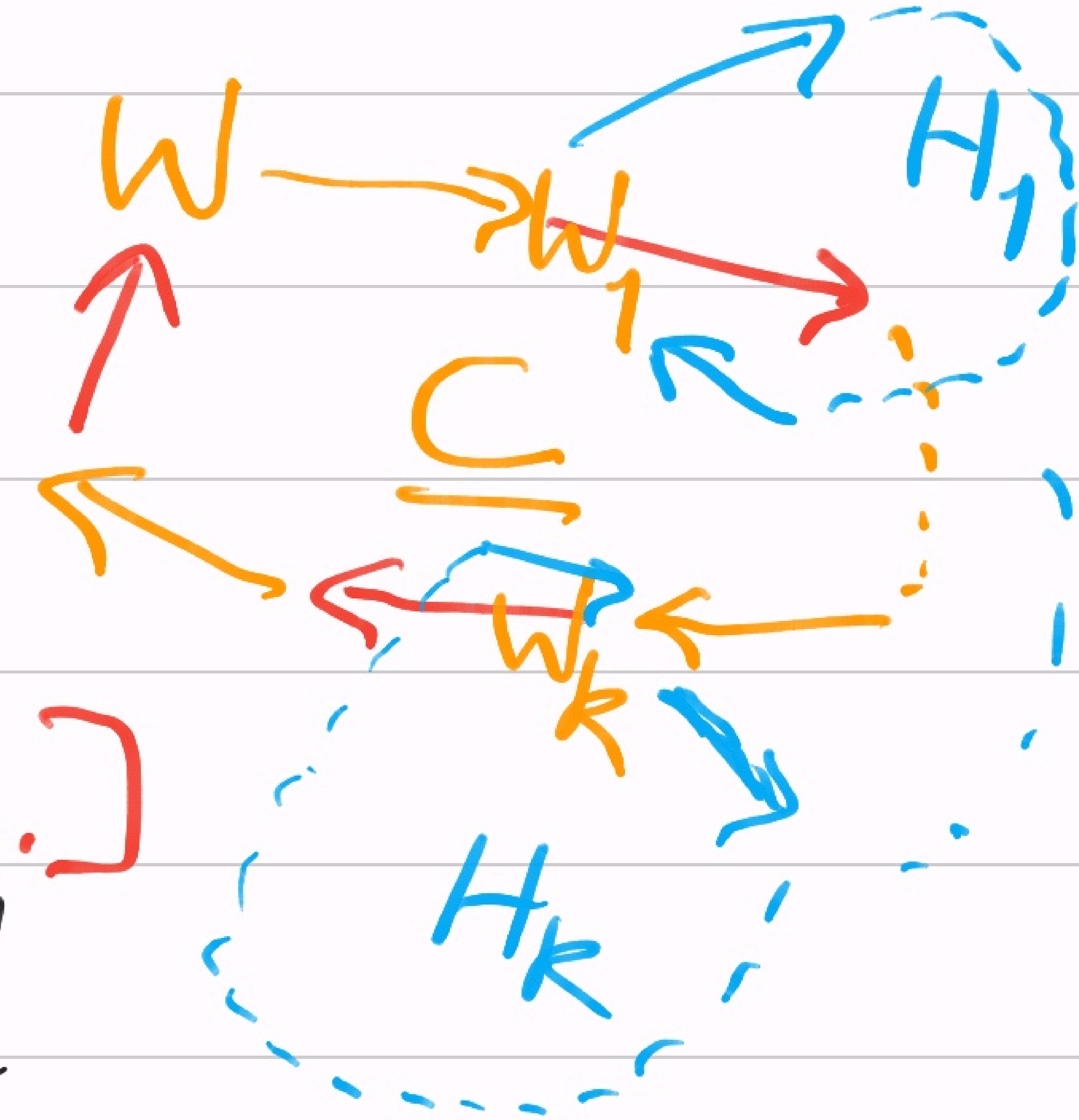
$\Rightarrow$ Wlog, we get a cycle $\underline{C}$.

$\Rightarrow$ either we're done.

or G has <u>unwalked</u> edges:

- Let <u>H</u> be <u>subgraph</u> <u>excluding</u> the cycle C edges. [H is even-deg.]
- Let $H_1, H_2, \ldots, H_k$ be the connected components of H.
- Let <u>w_i</u> be the <u>first</u> vertex in C that takes you to H_i.
- Use induction to find Eulerian circuit in H_1, starting from w_1. [H_1 has <u>even</u> degree.]

<u>Glue</u>: $w \overset{C}{\rightsquigarrow} w_1$ to $\text{Euler}(H_1)$ to $w_1 \overset{C}{\rightsquigarrow} w_2$.



and so on till $w_k \overset{c}{\leadsto} w$.

$\triangleright$ This is an Eulerian tour of G. $\square$

$\rightarrow$ Implement this as a **fast** algorithm.

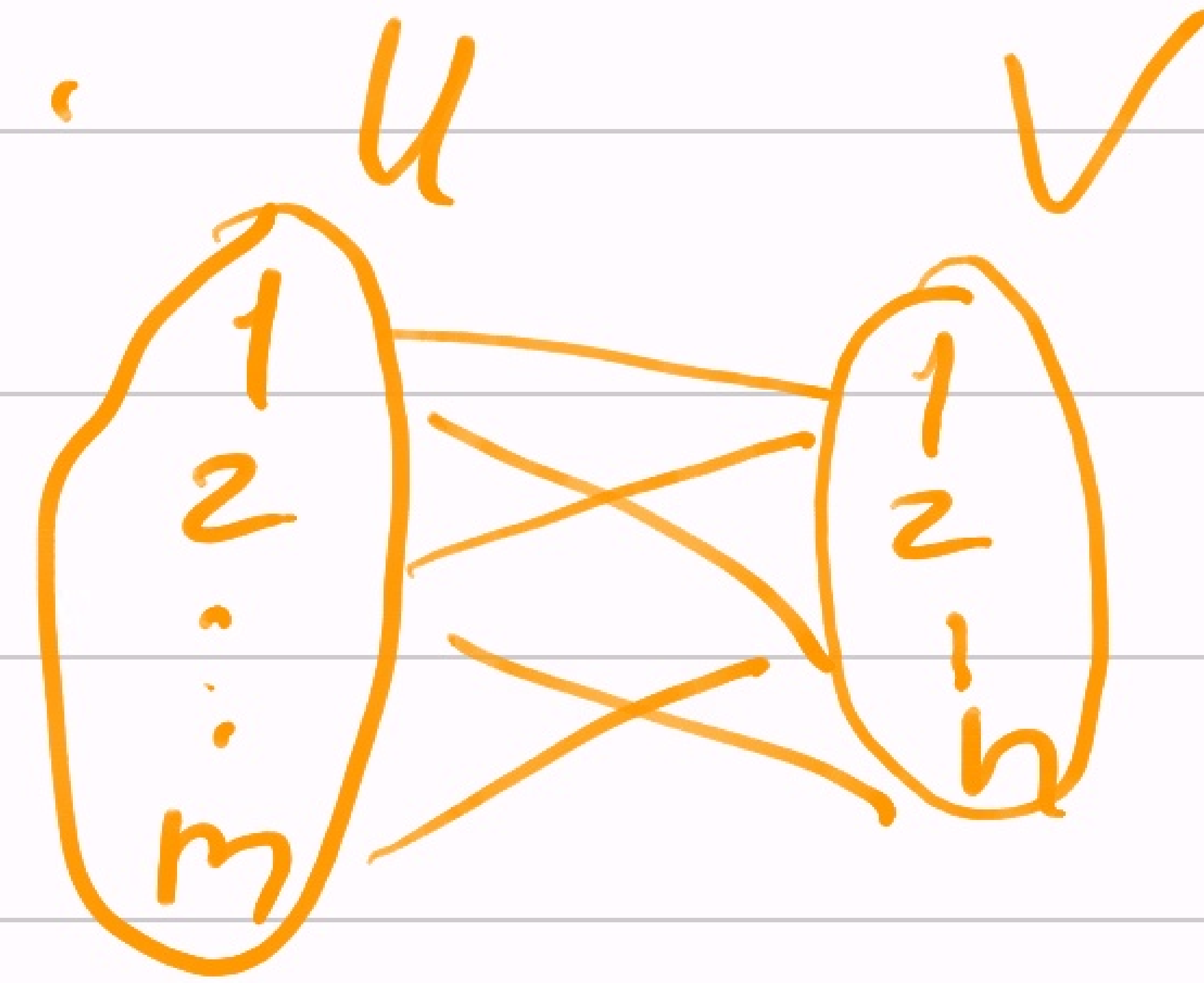
Hamiltonian path / circuit / cycle

- **Defn:** • A path $x_0, x_1, \dots, x_k$ is called a **Hamiltonian path** if it goes through all $V(G)$.
  • If $x_0, x_1, \dots, x_k = x_0$ goes through **all** $V(G)$, then it's a **Hamiltonian cycle**.

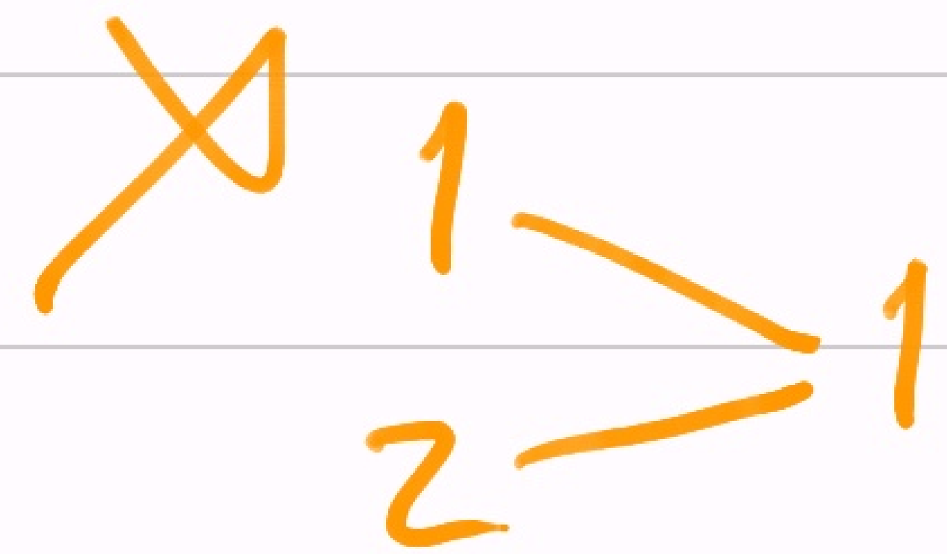
- Graph G with a Hamiltonian cycle is called a <u>Hamiltonian graph</u>.

– Egs. cycle C_n, complete K_n.

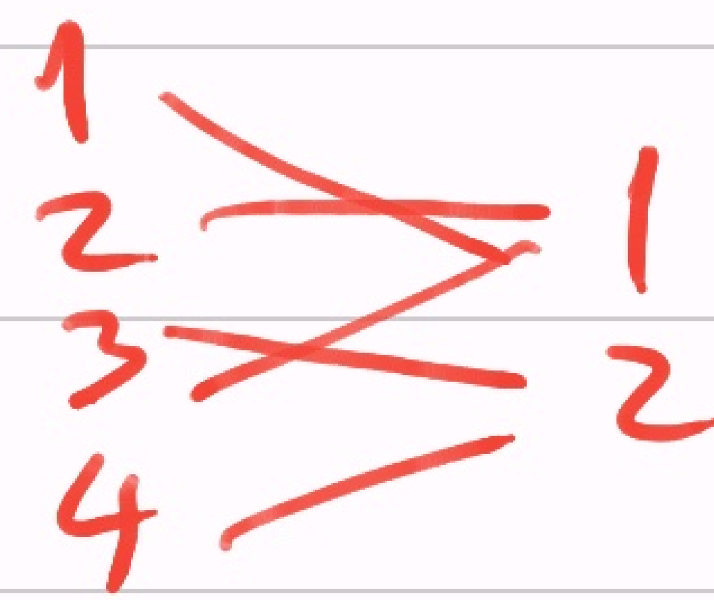
– $K_{m,n}$: Hamiltonian iff $1 < m = n$. (<u>Exercise</u>)



▷ Hamiltonian G remains so if we add edges.



▷ It has two disjoint paths $u \sim v$, $\forall$ vertices $u \neq v$.



[<u>biconnected</u> graph.]

▷ Finding HamPath is NP-complete. ⟶ (very hard)

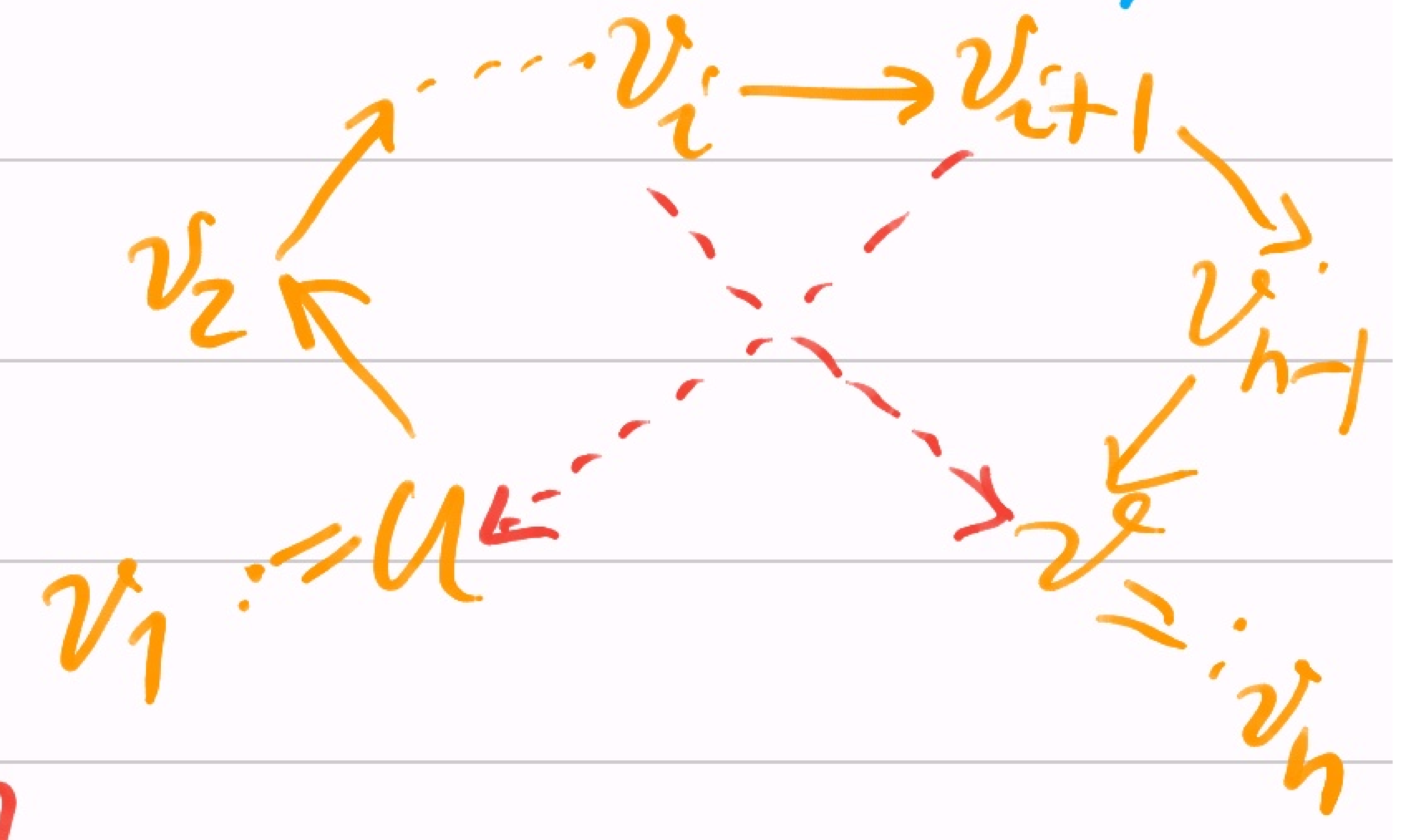
→ We don't have "necessary & sufficient" conditions for Hamiltonian graphs.

<u>Theorem</u> (Dirac, 1952): A simple graph on $n \geq 3$ vertices is Hamiltonian, if every vertex has degree $\geq n/2$.

<u>Pf</u>: • Suppose H is a graph with $\deg \geq n/2$, but it's <u>not</u> Hamiltonian.

• Keep on adding edges till it's non-Hamiltonian and call this graph $\underline{G}$. ($\triangleright$ G + (any $\underline{\text{new}}$ edge) is Hamiltonian)

$\Rightarrow$ Any $\underline{\text{non}}$-edge (u,v) has a Hamiltonian-path $v_1, v_2, \dots \to v_n$ (else edge (u,v) can be added to G!)



- Define $\underline{\text{neighbours}}$ of v_1 & v_n as:

$$S := \{\, i \mid (u, v_{i+1}) \in E(G) \,\}$$

$$T := \{\, i \mid (v_i, v) \in E(G) \,\}.$$

- Suppose $S \cap T$ has $\underline{\text{some}}$ i.

$\Rightarrow$ G has HamCycle, namely: $v_1, v_2, \to v_i, v, v_{n-1}, \dots v_{i+1}, v_1$. $\Rightarrow$ ↯ $\Rightarrow$ $S \cap T = \emptyset$.

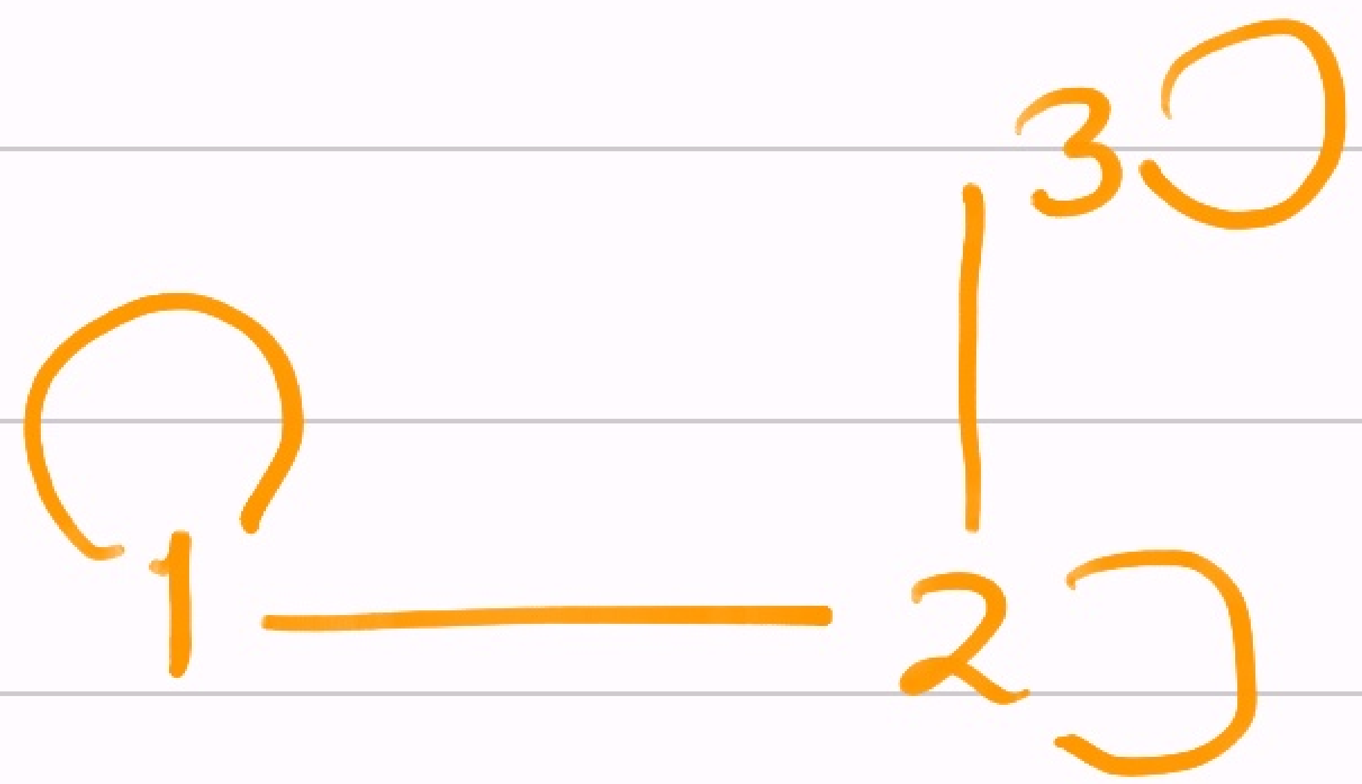
$\Rightarrow \quad S \cup T = [n] \quad (\text{as } |S|, |T| \geq \deg \geq \frac{n}{2})$

• But, $n \notin S \cup T$. $\Rightarrow$ ↯ ($\because$ G has no loop (v,v))

$\Rightarrow$ G is Hamiltonian

$\Rightarrow$ H " " . $\square$

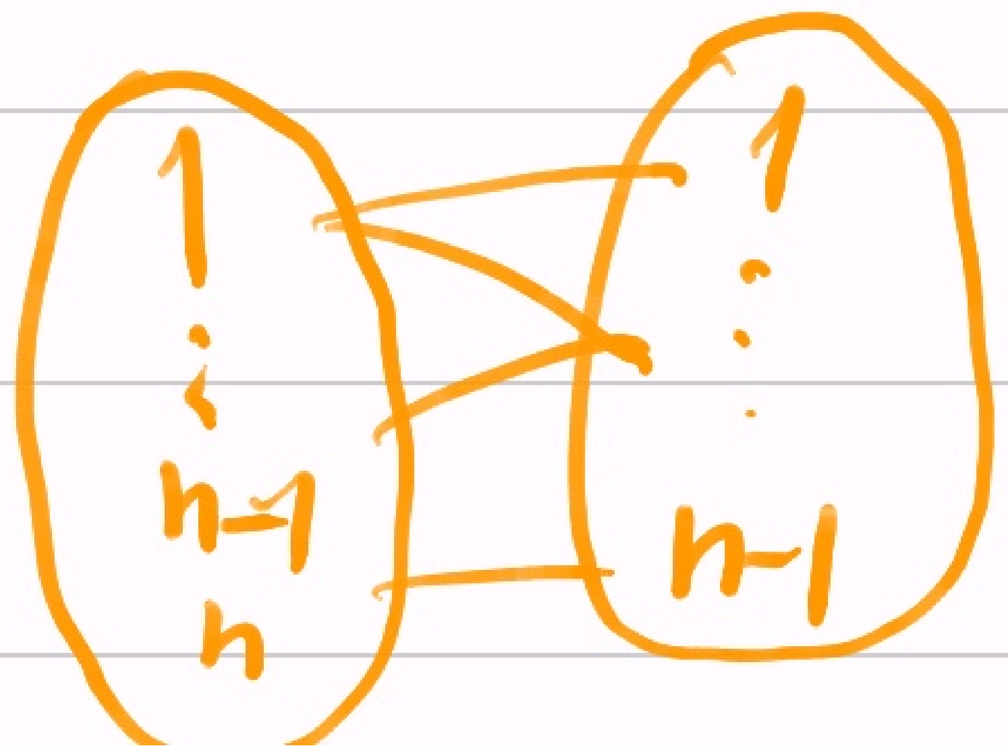
Eg1. H



$n=3$, $\frac{n}{2}=1.5$

$\deg=2$, but H is non-Ham.

Eg2. $K_{n,n-1}$:



$\deg = n-1 < \frac{2n-1}{2}$

& non-Hamiltonian.

- Eg. 3: C_n has deg $= 2$, but Hamiltonian.

$\Rightarrow$ Dirac prop. is tight (i.e. sufficient condn.) but it's <u>not</u> necessary.

- More combinatorial properties of graphs: coloring, independent set, vertex cover, etc.

$\to$ Coloring a graph you've seen examples.

<u>Qn:</u> Biggest set of <u>mono</u>chromatic vertices?

Independent Set

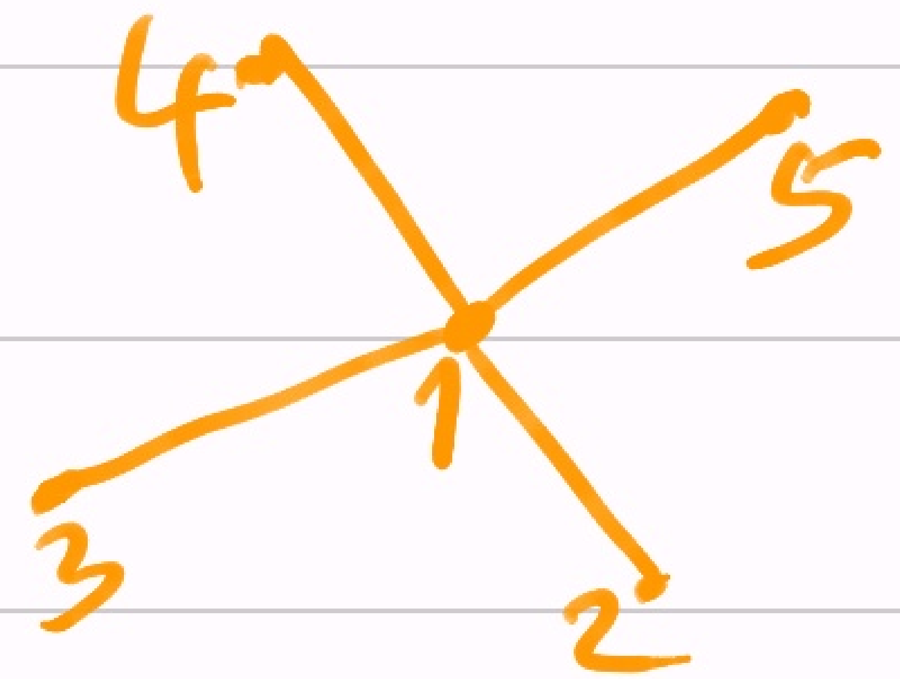
- Eg. Relationship graph of people.

- **Defn**: • An independent set of $G=(V,E)$ is a subset $S \subseteq V$ s.t. $\forall\, u,v \in S : (u,v) \notin E(G)$
(or stable set)
• Stability number $\alpha(G)$ is the size of a max. indep. set S in G.
$(\, 1 \leq \alpha(G) \leq |V| = n \,)$

Qn: Fast algo. for $\alpha(G)$? (NP-complete)

• <u>Maximal indep. set</u> is an indep. set S s.t. $\forall v \in V \setminus S$, $S \cup \{v\}$ is <u>not</u> indep.

- Eg. G:



$\Rightarrow \alpha(G) = 4$.

▷ $\{1\}$ is maximal indep. set.

(unique?)

- Recall $\overline{G} :=$ complement of graph $G = (V, E)$.

$= (V, \overline{E})$

▷ $|\overline{E}| + |E| = \binom{|V|}{2} = \binom{n}{2}$.

▷ IS S of $G \iff$ In $\overline{G}$, S is complete graph.

- Defn: Clique S of graph G has all possible edges. (It gives a complete subgraph of G.)

Exercise: Maximal indep. set is easy to find, in a given graph.

- Idea: Pick a vertex v. $S \leftarrow S \cup \{v\}$.
  Remove the edges of v from G.
  Repeat (& grow S).

- Defn: • Vertex Cover $S \subseteq V(G)$ is a subset s.t.
  $\forall (u,v) \in E(G)$, $(u \in S$ OR $v \in S)$.

- Least vertex cover of G is one where $|S|$ is min.

▷ 2-factor approximate vertex cover is easy to find.

Pf:
- Pick $(u,v) \in E(G)$. $S \leftarrow S \cup \{u,v\}$.
- Remove edges incident on u or v.
- Repeat (& grow S).

Exercise: Prove that $|S| \leq 2 \times (\text{least}\,|VC|)$.

$\square$

Qn: Better approx. are hard problems?

▷ S is a vertex cover $\iff$ $V-S$ is an independent set. [Pf: $\bar{S}$ has no edge. □]

▷ Thus, least vertex cover & maximum indep. set & maximum clique are all equivalent problems!

Qn: Approximate indep. set is hard?