Basic Graph Theory

- Graphs provide a natural way to model connections between objects.
  - Eg. communication networks, social networks, transport network, etc.

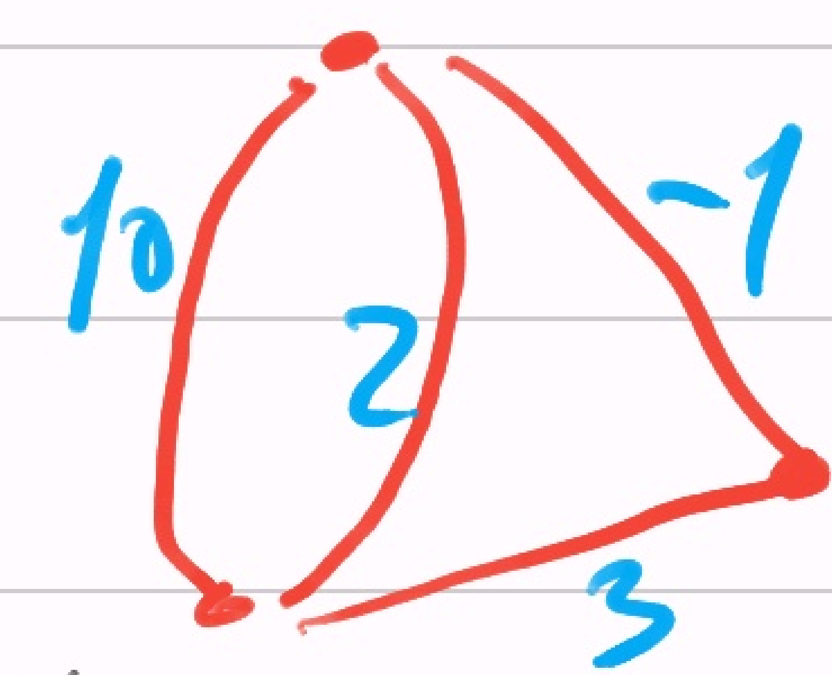
- Defn: • Graph $G=(V,E)$ has vertices V & edges $E \subset V \times V$ s.t. $(u,v) \in E$ means that u is connected to v, or v is adjacent to u.
  • $V(G)$ or $E(G)$.

- G is <u>simple</u> if E has no <u>loop</u>, i.e. $(u,u) \notin E$.

▷ Simple graph on n vertices can have $0 \leq |E| \leq n(n-1)$ [or $\binom{n}{2}$ if G is <u>undirected</u>]

— If two vertices have <u>multiple</u> edges, then G is called a <u>multigraph</u>.

— Edges with directions $\rightarrow$ <u>directed</u> G.
" in both " $\rightarrow$ <u>undirected</u>.



— Edges with <u>real</u> weights $\rightarrow$ <u>weighted</u> graph.

- Egs. of graphs:

1) Model social networks: V = members & E = relationship between members.

eg. friendship ⇒ undirected $G=(V,E)$

"follows" ⇒ directed "

"is senior/older" ⇒ " "

2) How processes run in a computer.

eg. Process-a can be run only if Process-b completes. Precedence graph: V = processes

$E \ni (u,v)$ if u should run before v.

3) Schedule exam for different courses, where courses may're common students. #Time slots?

$G=(V,E)$ with $V=$ courses & $E \ni (u,v)$ if u & v have ≥ 1 common student.

$\triangleright$ #time-slots $=$ #colors needed to color the graph (s.t. no edge is monochrome)

4) Color the states in a map of India (s.t. no adjacent states get the same color)

5) Road network is a weighted-edge, directed graph. Shortest-distance (path) between $s \to t$.

- **Defn:** • Subgraph $H = (V, E')$ of $G = (V, E)$ if $E' \subseteq E$. (Some edges of E may not be in E')

• Degree of a vertex v (in G) is the number of edges with v. $\triangleright$ $0 \leq d(v) \leq |V| - 1$.

Theorem: $$\sum_{v \in V} d(v) = 2 \cdot \#\text{edges}.$$

$\Rightarrow$ # vertices with odd-degree = even.

- Let's see some named graphs:

1) Complete graph (K_n): all $\binom{n}{2}$ edges on $V=[n]$.

$\triangleright$ $d(v) = n-1$.

2) Path graph (line): $v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_n$.

3) Cycle graph (C_n): $v_1 \longrightarrow v_2 \cdots \longrightarrow v_n \longrightarrow v_1$

$\triangleright$ $d(v)=2$

Ex.1: $\forall v,\ d(v)=2 \Rightarrow G = \sqcup(\text{cycle})$.

Ex.2: $\forall v,\ d(v)\leq 2 \Rightarrow G = \sqcup \text{ cycle} \sqcup \text{line} \sqcup \text{point}$.

4) Bipartite graph: $V = S \sqcup T$ s.t. $E \subseteq S \times T$.

Complete " " ($K_{m,n}$): $E(K_{m,n}) = [m]\times[n]$.

5) Regular graphs: G is d-regular if $\forall v \in V(G)$, $d(v) = d$.

Ex.: # edges on d-regular graph on $[n]$ $= dn/2$.

$\Rightarrow$ 5-regular graph on 101 vertices doesn't exist.

Representation of graphs

1) Adjacent list of the graph as linked-lists.

2) Adjacency matrix: $A_G := ((1 \text{ iff } (u,v) \in E(G) \mid (u,v)))$

$V \times V \to \{0,1\}$.

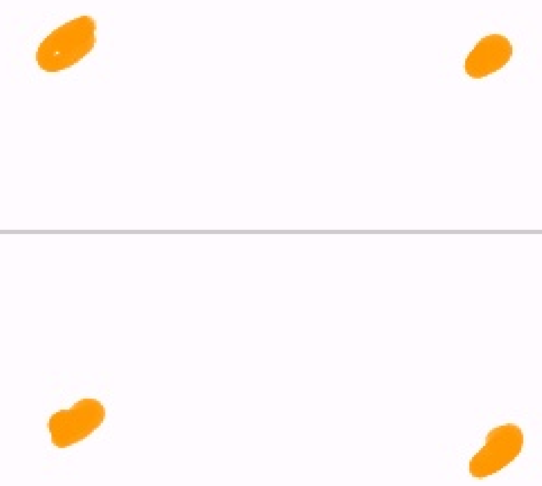
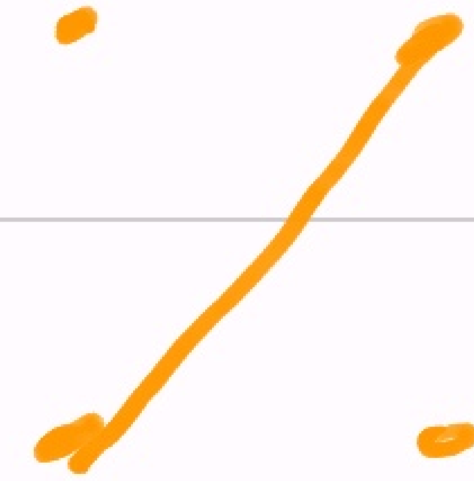
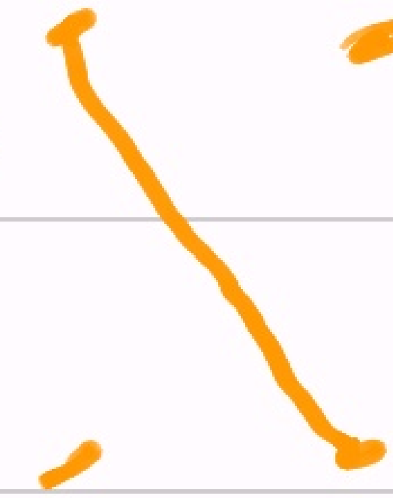
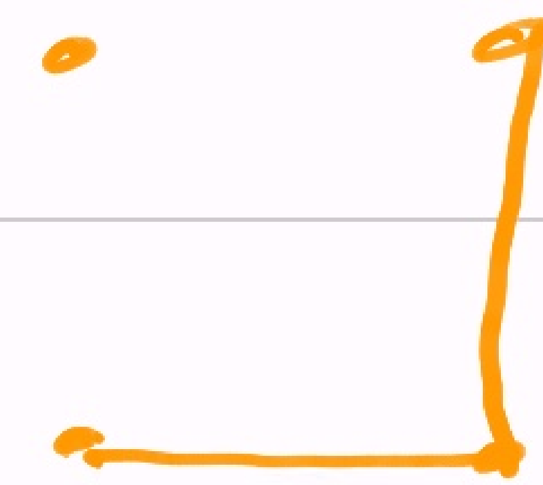
3) Incidence matrix : $M_G := \big((1 \text{ if } v \in e \mid (v,e) \in V \times E) \big)$

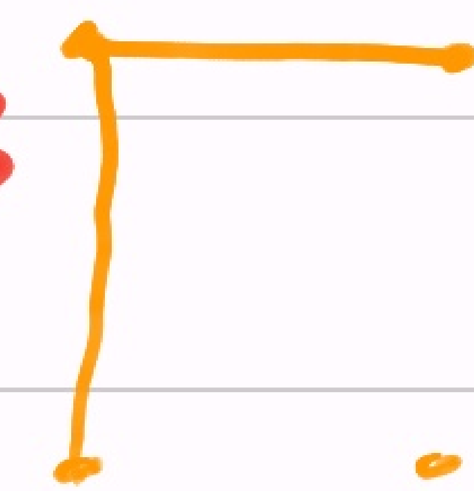
$V \times E \to \{0,1\}$

$\triangleright \quad M_G \cdot M_G^T = A_G$ Pf: Ex. $\square$

Graph isomorphism

- Defn: Graphs G & H are isomorphic, if $\exists$ bijection $\tau : V(G) \to V(H)$ s.t. $\forall (u,v) \in V(G)^2$: $(u,v) \in E(G)$ iff $(\tau(u), \tau(v)) \in E(H)$.

H_1  H_2  H_2  H_3 

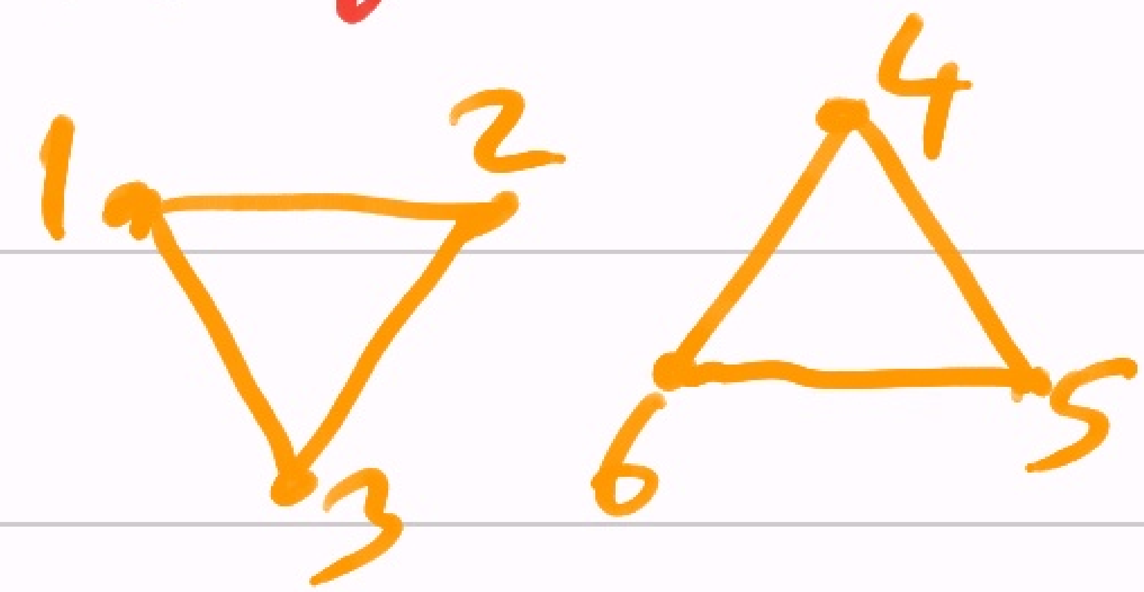
H_3 

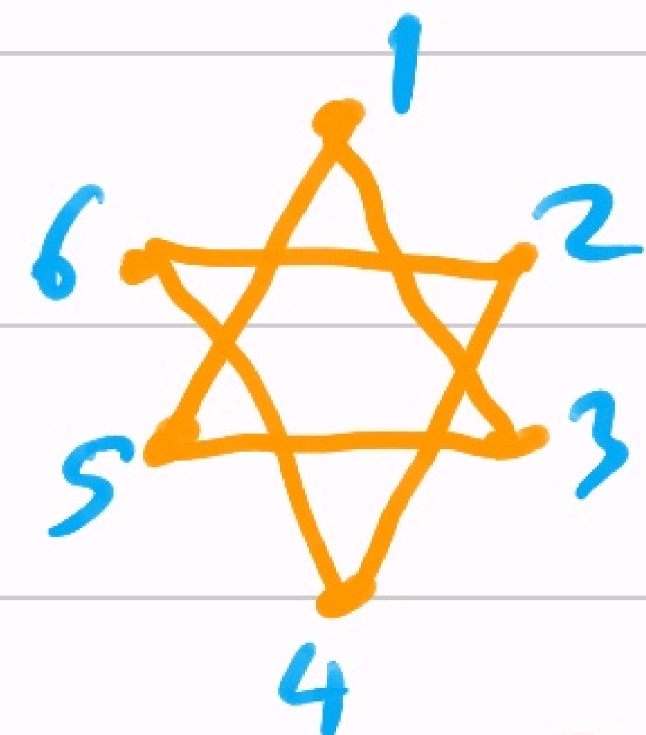
- For graph G, complement $\overline{G}$ is the graph with $E(\overline{G}) =: \{(u,v) \mid (u,v) \notin E(G)\}$.

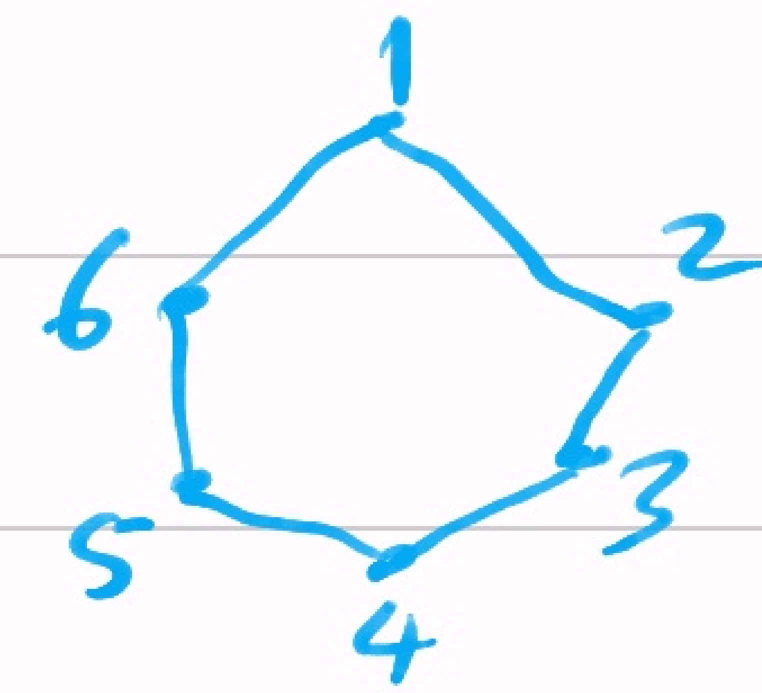
▷ $G \cong H$ iff $\overline{G} \cong \overline{H}$.

Qn: Construct all non-isomorphic graphs on $|V|=4$ & #edges ≤ 4.

Qn: Non-isomorphic graphs but same degree sequence?

G' 

G 

H 

$C_3 \sqcup C_3$ $\qquad$ C_6

- Adjacency matrix A_G (resp. $A_{G'}$) depends on the <u>order</u> of the vertices.

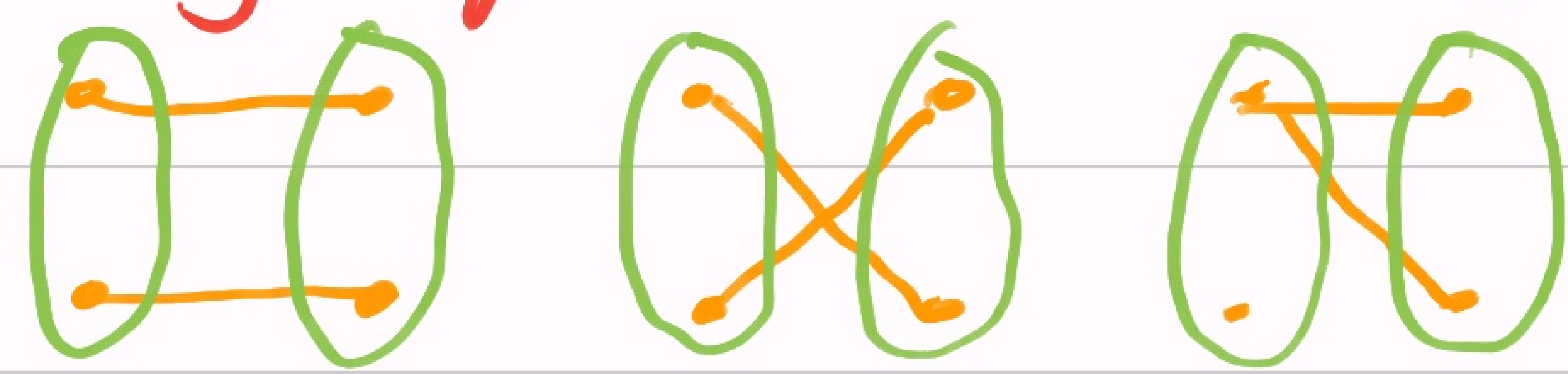
  Qn: $\exists$? permutation π on row & col that makes $\pi(A_G) = A_{G'}$?

  (<u>Read</u>: Babai's algo. for GI.)

- **Defn:** Graphs G, T are related if $G \cong T$ (isomorphic).

▷ Isomorphism is an equiv. relation.
Each class contains the isomorphic graphs.

- **Qn:** • non-isomorphic bipartite graphs on $m+n$ vertices?



• Two d-regular graphs that are non-isomorphic?

Connectivity

- **Defn:** • Walk of length k is a sequence of vertices $x_0, x_1, \dots, x_k$ where $(x_i, x_{i+1}) \in E$, for $0 \le i \le k-1$. (except may be x_0, x_k)
  • If all the vertices are distinct, then the walk is called path. (or simple path)
  • If $x_0 = x_k$ & length > 2, then the path is called a cycle.

▷ walk $\supsetneq$ path $\supsetneq$ cycle. eg. 1, 2, 3, 2, 4 is a walk; not path.

Eg. $V = [4]$, $E =$ 

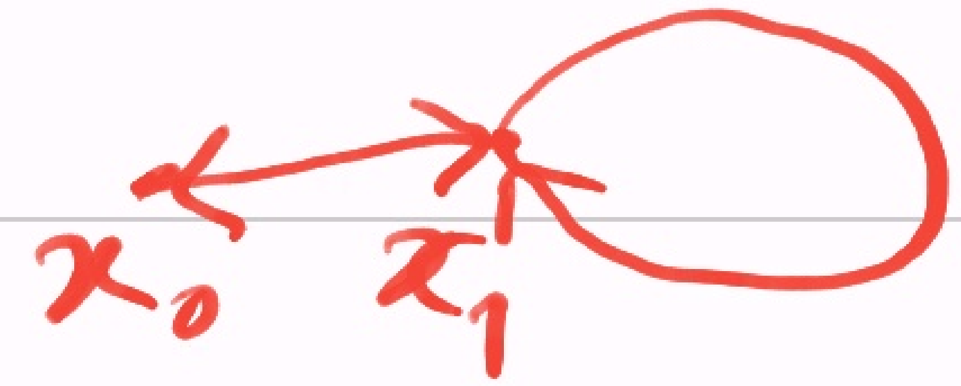
- $(1,2,3,4,1)$ is walk, path, cycle.
- $(1,2,4,3,2,1)$ is walk, not path.
- $(1,2,3,4)$ is walk, path, not cycle.

Lemma 1: Walk between u,v $\Longrightarrow$ Path between u,v.

Pf:
- Consider walk $\underline{P}$: $u =: x_0, x_1, \to x_k =: v$ of least length. If x_i's are distinct, then done!
- Let v occur twice in P. Delete the section from $v \rightsquigarrow v$ in P. We get a walk $x_0 \rightsquigarrow x_k$

of smaller length $\Rightarrow$ ↯ $\square$

$\Rightarrow$ P is path.



<u>Lemma 2:</u> Walk $x_0 \rightsquigarrow x_k$ with $x_0 = x_k$ & no two consecutive edges the same $\Rightarrow$ there is a cycle.

<u>Pf:</u>
- Shortest walk <u>P</u>: $x_0 \rightsquigarrow x_k$ is a path.
- If $|P| \geq 3$, then use induction on length.
- If $|P| < 3$: $|P| = 2 \Rightarrow x_0, x_1, x_2 = x_0$ $\Rightarrow$ consecutive edges same $\Rightarrow$ ↯ $\square$

1 — 2

e.g. 1, 2, 1

(length = 2)

path / <u>not</u> cycle

Defn: • Graph G is connected, if $\forall u \neq v \in V$ $\exists$ path $u \rightsquigarrow v$.

• Vertices u, v are related if $\exists$ path $u \rightsquigarrow v$.

▷ Connectivity is an equiv. relation.

– This gives classes, called connected components of the graph G.

▷ $1 \leq \#$ components $\leq |V|$.

▷ Say, vertex-1 is connected to every vertex $\Rightarrow$

graph is connected!

- Consider graph G & its adjacency matrix A.

- Qn: What about <u>matrix product</u> A^2?

$\triangleright$ $(A^2)_{u,v} \neq 0$ iff $\exists$ path $u \rightsquigarrow v$ of length ≤ 2 in G.

Pf: $(A^2)_{u,v} = \sum_{w \in V} A_{u,w} \cdot A_{w,v} > 0$ iff $\exists$ $u - w - v$ in G. $\square$

$\triangleright$ $(A^2)_{u,v} \geq$ # paths $u \rightsquigarrow v$ of length ≤ 2.
[≥ 1 if $(A^k)_{u,v} \neq 0$.]

- Lemma: $(A^k)_{u,v} =$ # walks $u \rightsquigarrow v$ of length-k in G.

Fix $A_{u,u} := 1$.

Pf: $(A^k)_{u,v} > 0$ due to the contributions by length-k paths of the type:

$$u =: v_0 \to v_1 \to v_2 \cdots \to v_{k-1} \to v_k =: v$$

Ex: Each such walks contributes a $\underline{\underline{1}}$ in $(A^k)_{u,v}$.

$\square$

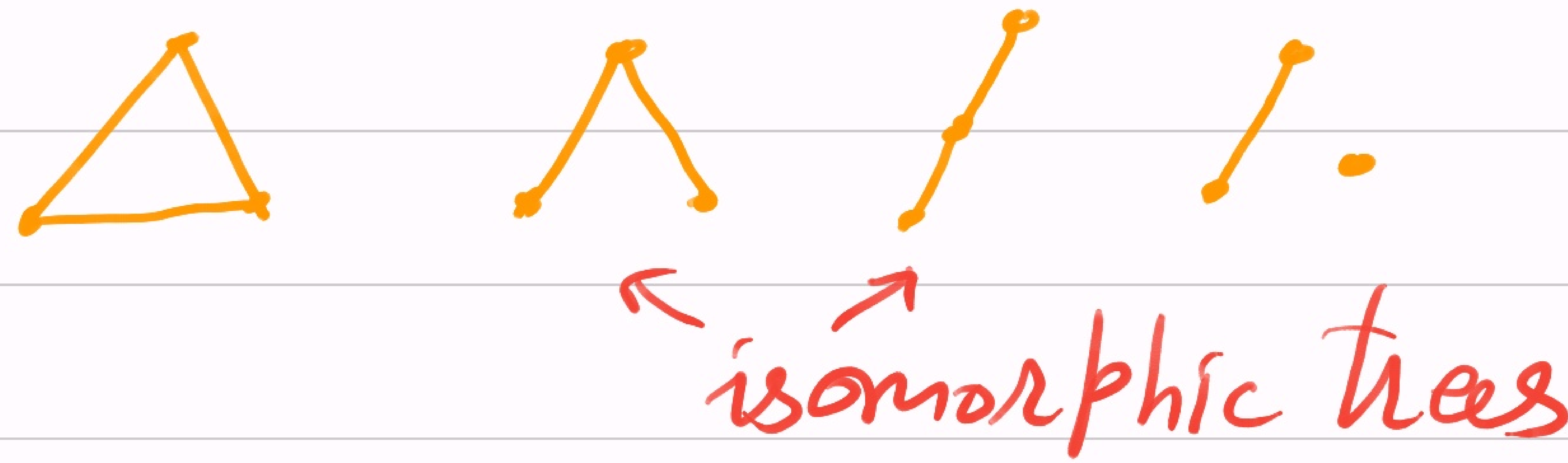
Corollary: Say $A_{u,u} = 0$. $(A^k)_{u,v} =$ # walks $u \rightsquigarrow v$ of length-k.

Corollary: $(A^k)_{u,v} \geq$ # paths $u \rightsquigarrow v$ of length $\leq k$.
[≥ 1 if $(A^k)_{u,v} \neq 0$]

Trees

Defn: Tree is a connected graph with no cycle.

- Eg. of a tree & non-tree on $n=3$:



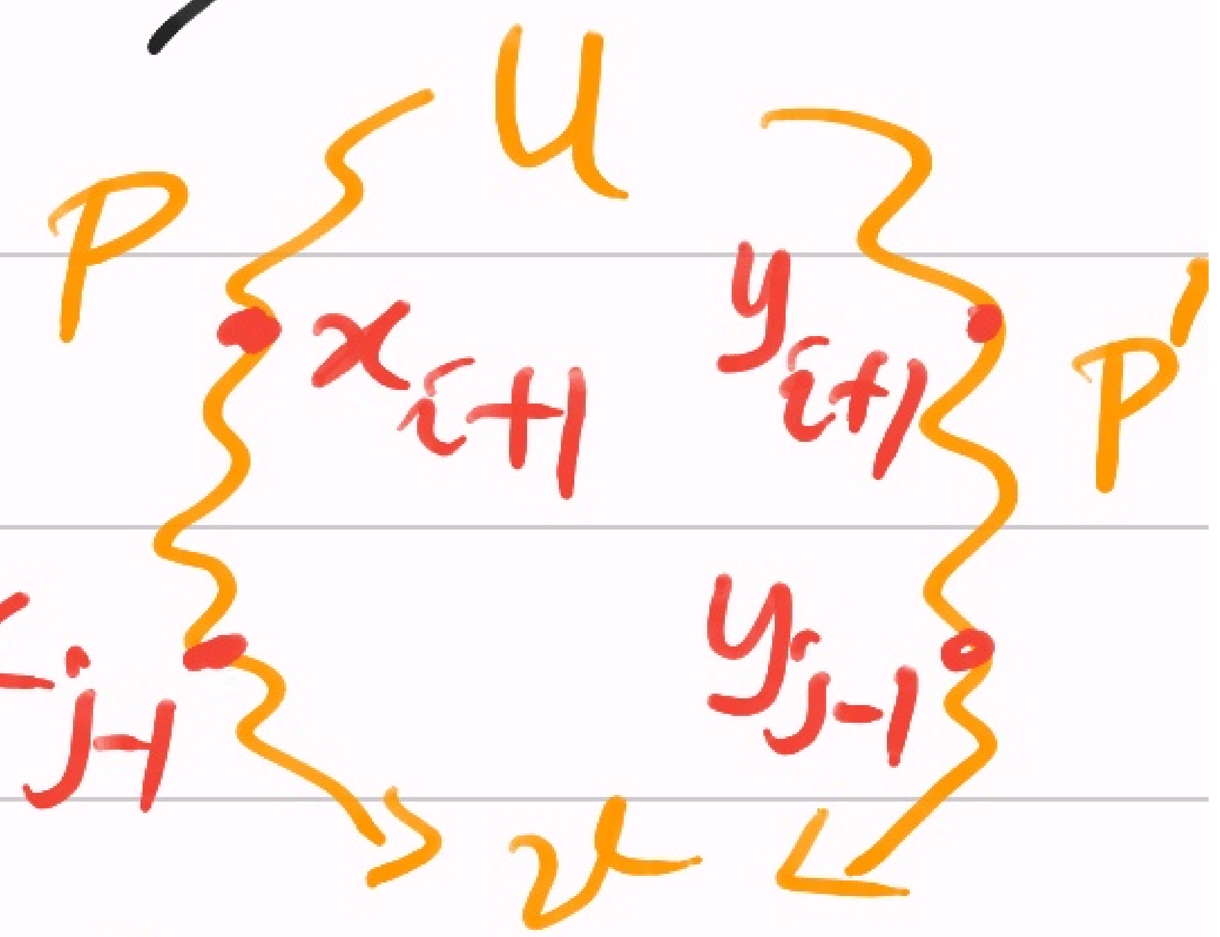
Theorem: G is a tree iff $\forall\, u, v \in V(G)$, there is a unique path $u \rightsquigarrow v$.

Pf: ($\Rightarrow$): Let G be a tree. Assume two paths between u, v are:

$$P =: \{\, u = x_0 \to x_1 \to x_2 \cdots \to x_k = v \,\}$$
$$P' =: \{\, u = y_0 \to y_1 \to y_2 \cdots \to y_k = v \,\}$$

(Wlog equal length)

- Least i : $x_{i+1} \neq y_{i+1}$ $\triangleright$ $j - i \geq 2$.
- Max j : $x_{j-1} \neq y_{j-1}$.



- Consider the walk

$$x_i, \dots, x_j = y_j, y_{j-1}, \dots, y_i = x_i.$$

It has no consecutive edges the same ($\because$ P, P' are paths).

$\Rightarrow$ $\exists$ cycle in G $\Rightarrow$ $\lightning$

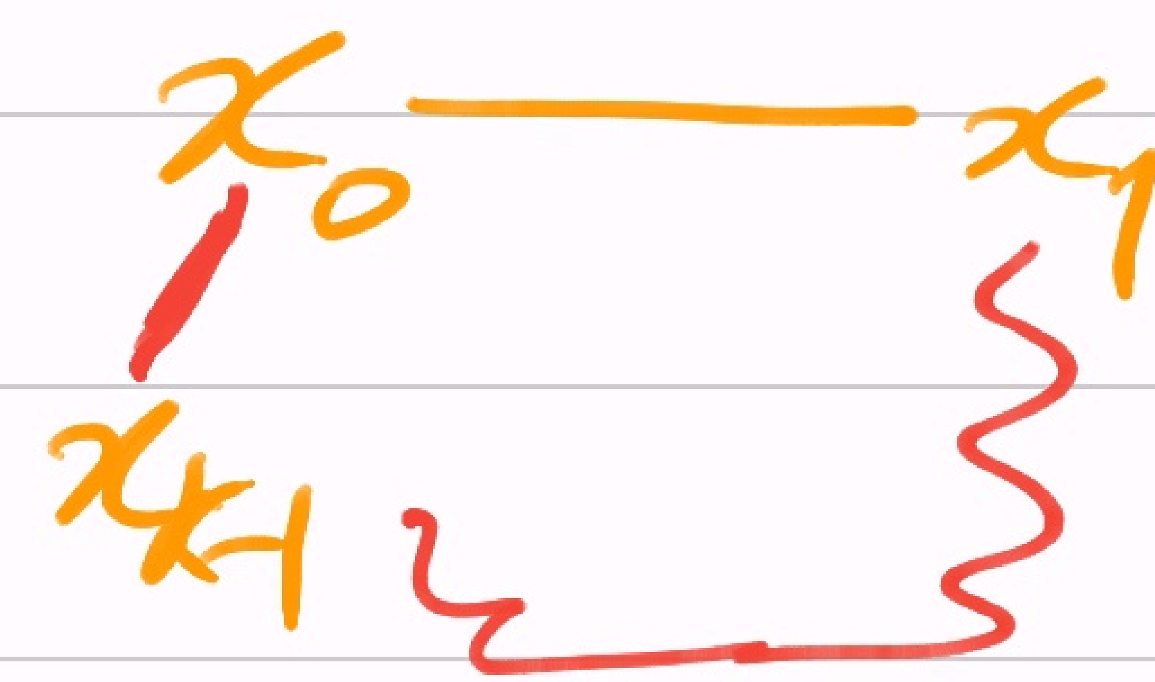
$\Rightarrow$ $u \rightsquigarrow v$ path is <u>unique</u>.

$\boxed{\Leftarrow}$: Assume $\forall\, u, v$, the path is unique.

$\Rightarrow$ G is connected.

- Suppose G has a <u>cycle</u>: $x_0 \to x_1 \to \cdots \to x_k = x_0$.

$\Rightarrow$ $x_0 \rightsquigarrow x_1$ has <u>two</u> distinct paths $\Rightarrow$ $\lightning$



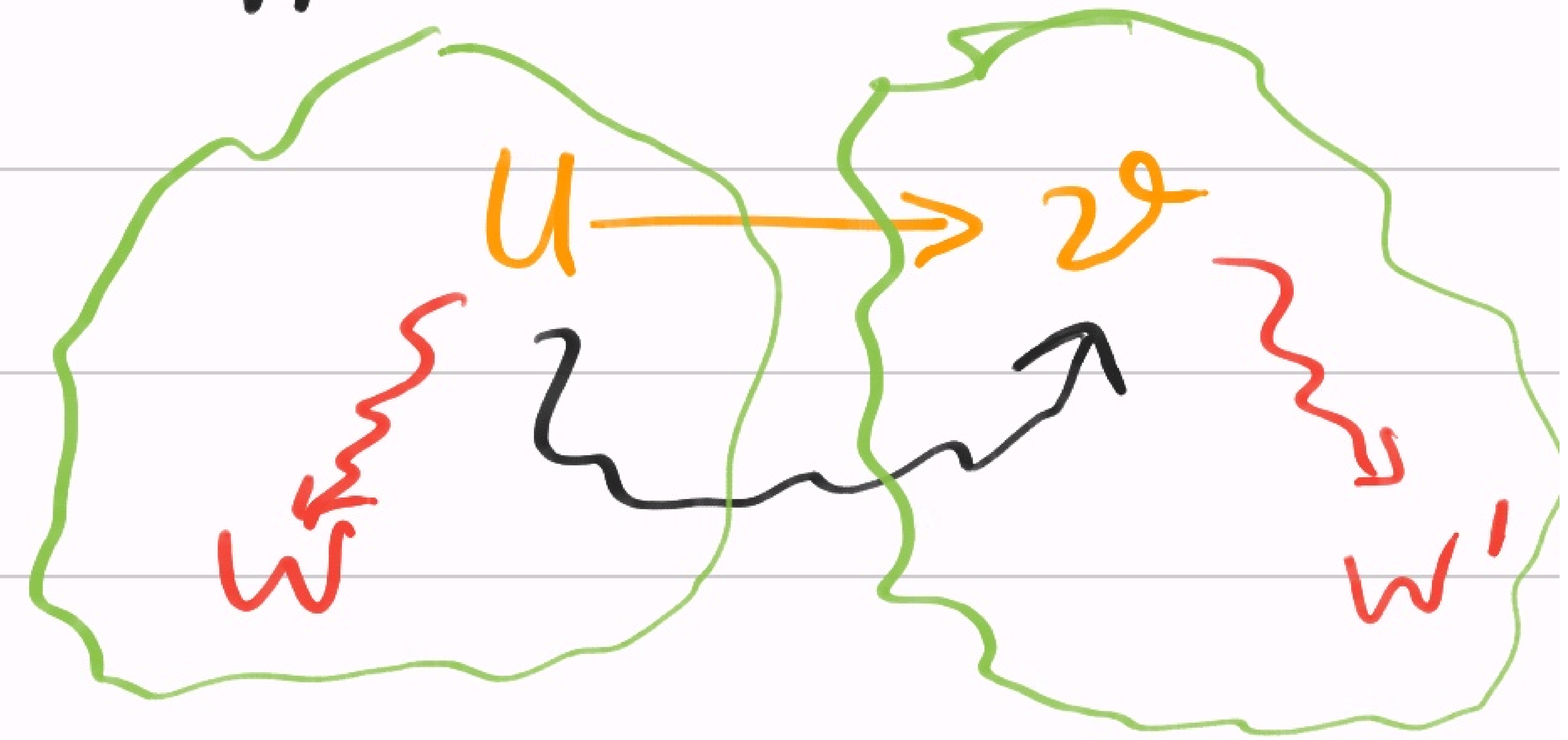
$\Rightarrow$ G is a <u>tree</u>.

$\square$

$\triangleright$ From a tree, removing an edge (u, v) <u>disconnects</u> u from v! ($\Rightarrow$ <u>Forest</u>)

Pf: Let $G' := G \setminus (u,v)$. Suppose G' has a path $u \sim v$.

$\Rightarrow$ In G, $u \sim v$ path is **not** unique. $\Rightarrow$ ↯



$\Rightarrow$ In G', $\nexists\ u \sim v$.

and G' has **two** connected components; each is a **tree**. $\square$

▷ A **finite** tree has a degree one vertex.

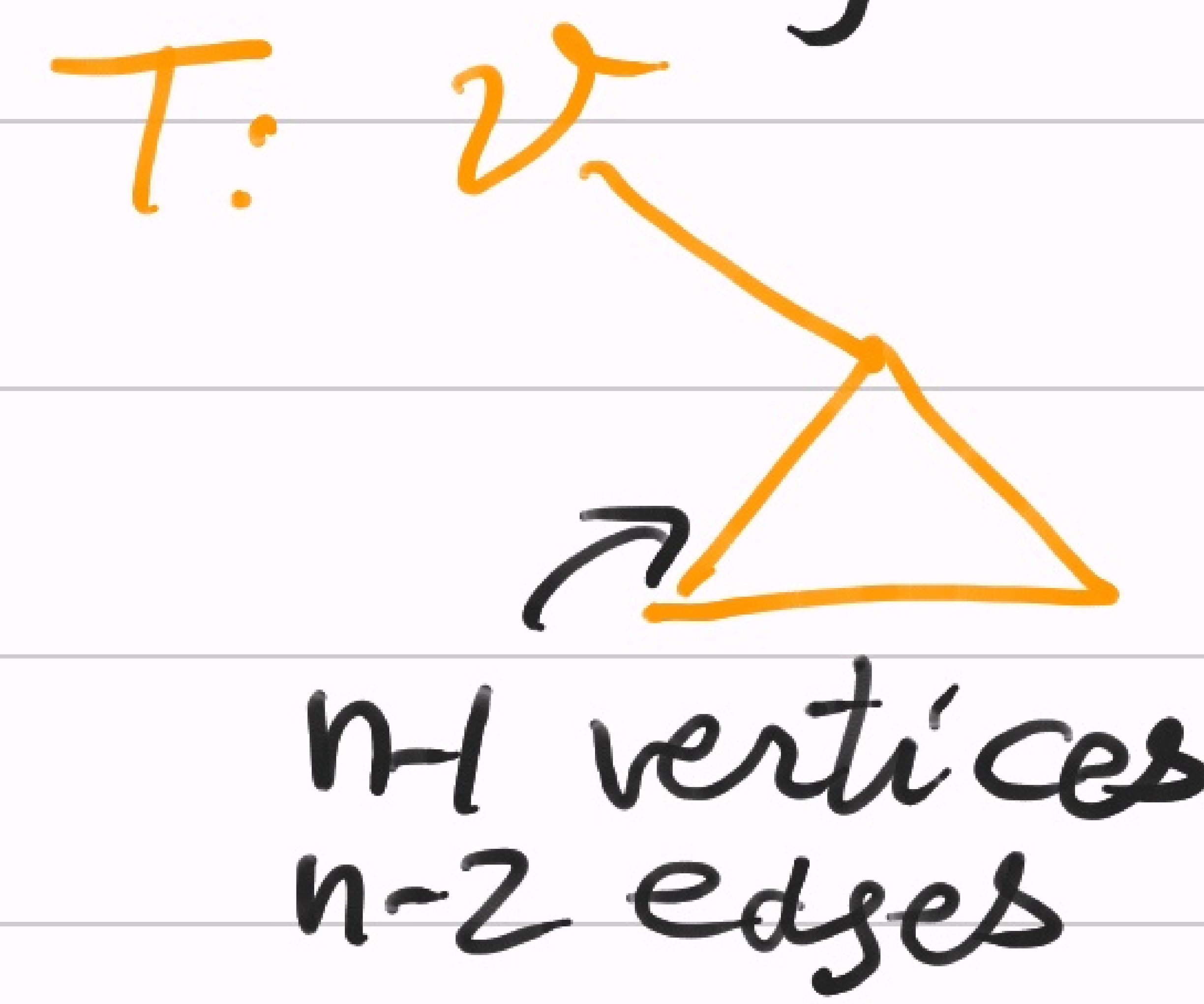
Pf: • $\deg \geq 2$ in a vertex allows to develop an **infinite** walk: $\rightsquigarrow \xrightarrow{e_1} v \xrightarrow{e_2 \neq e_1} \rightsquigarrow$

• This creates an infinite walk, which can't have a cycle (as we're walking in a tree). $\Longrightarrow \lightning$

$\Longrightarrow$ $\exists$ vertex of deg $= 1$. $\square$

Theorem: Tree on n vertices has $n-1$ edges.

Pf: • Let T be a tree & $v \in E(T)$ of degree $= 1$. (Base: $n=1$.)



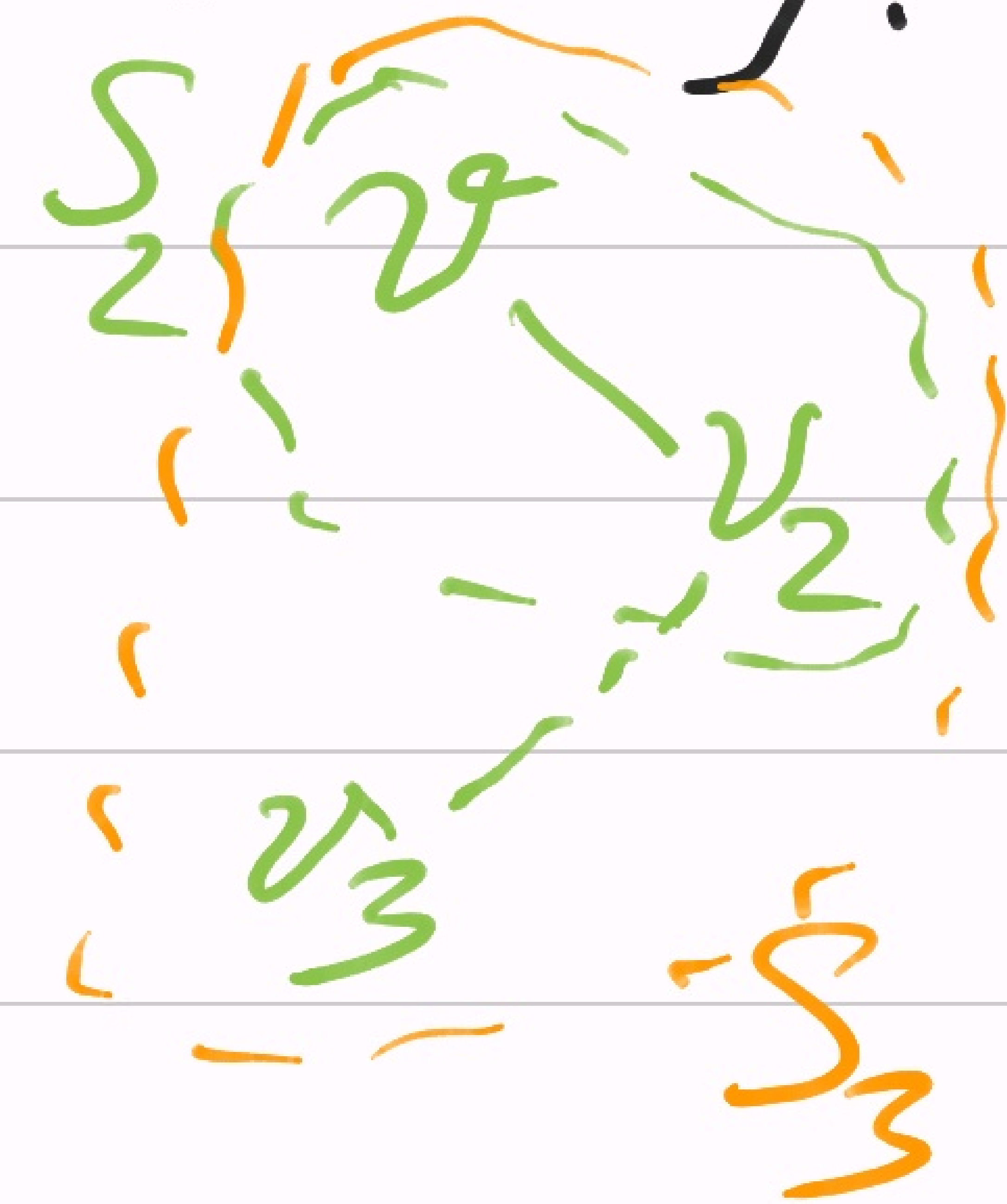
• From the subtree adjacent to v, we get (by induction):

$$|E(T)| = 1 + (n-2) = n-1. \qquad \square$$

- **Spanning tree**: For graph $G=(V,E)$, a subgraph T is called a spanning tree, if
  1) T is tree, &
  2) $V(T) = V(G)$. (Covers all vertices)

- **Algo (for connected G)**: Develop it iteratively.
  1) Let $S_1 =: \{v\}$.
  2) Let $S_2 \leftarrow S_1 \cup \text{neighbor}(v)$
  3) Let $S_3 \leftarrow S_2 \cup \text{neighbor}(S_2)$

  — and so on, to get S_n.



Ex: S_n is a spanning tree of G.