<u>Setting:</u> To count necklaces group D_n acts on n beads.

- Group G <u>acts on</u> set A as $G \ni g: A \longrightarrow A$, $a \mapsto g(a)$

[eg. $G = D_n$ acts on $A :=$ n-gon.]

- <u>Orbit</u> of $x \in A$ is $\underline{x^G} := \{ g(x) \mid g \in G \}$
- <u>Stabilizer</u> of $x \in A$ is $\underline{G_x} := \{ g \in G \mid g(x) = x \}$.

▷ $G_x \leq G$. ▷ $|G/G_x| = |x^G|$.

Pf: · Let $y \in x^G$ & $g: x \mapsto y$.

· Associate y with the coset $g \cdot G_x$.

$\Rightarrow \forall h \in g.G_x$, $h(x) = g(x) = y$.

$\Rightarrow$ We're <u>bijection</u> $x^G \longrightarrow G/G_x$. $\square$

- Fixed points of $g \in G$: $A_g := \{x \in A \mid g(x) = x\}$.

– Now, let's count the # necklaces (c colors, n beads)

$G := D_n$; $A := [c]^{[n]}$ = c-colorings on n-gon vertices.

▷ Necklace $N \in A$ is $N : [n] \to [c]$.

– The setting we're interested in is: $|G| \ll |A| = c^n$
$\overset{\shortparallel}{2n}$

- For $g \in G$, $n_g := \#$(cycles in g written in cycle-representation)

– Eg. $n_{(123)(4)} = 2$ for $n = 4$.

▷ $\forall g \in G$, $|A_g| = c^{n_g}$. Pf: $N \in A$ is moved by g iff N colors some cycle in g non-monochromatic.

▷ # distinct necklaces = # G-orbits in A

$$= \sum_{x \in A} \frac{1}{|x^G|} = \sum_{\text{"distinct" } x \in A} |x^G| \cdot \frac{1}{|x^G|} = \sum_{\substack{\text{distinct} \\ \text{necklaces}}} 1 .$$

$$= \frac{1}{|G|} \cdot \sum_{x \in A} |G_x| \quad = \text{(\# 1's in the matrix below)} / |G| .$$

- Goal: Rewrite this as $\sum_{g \in G} (\cdot)$?

- The (i,j)-th entry of the matrix is: $\begin{cases} 1, & \text{if } g_i(x_j) = x_j \\ 0, & \text{else.} \end{cases}$

$$\begin{array}{c|cccc} & x_1 & x_2 & \cdots & x_{|A|} \\ \hline g_1 & & & & \\ g_2 & & & & \\ \vdots & & & & \boxed{1/0} \\ g_{|G|} & & & & \end{array} \qquad A \longrightarrow, \quad G \downarrow$$

▷ $\#\,G\text{-orbits in } A = \dfrac{\sum_{x\in A}|G_x|}{|G|} = \dfrac{\sum_{g\in G}|A_g|}{|G|}.$

[Burnside's Lemma / Orbit-counting]

- Necklace application:

Thm [Polya's Enumeration]: # distinct necklaces

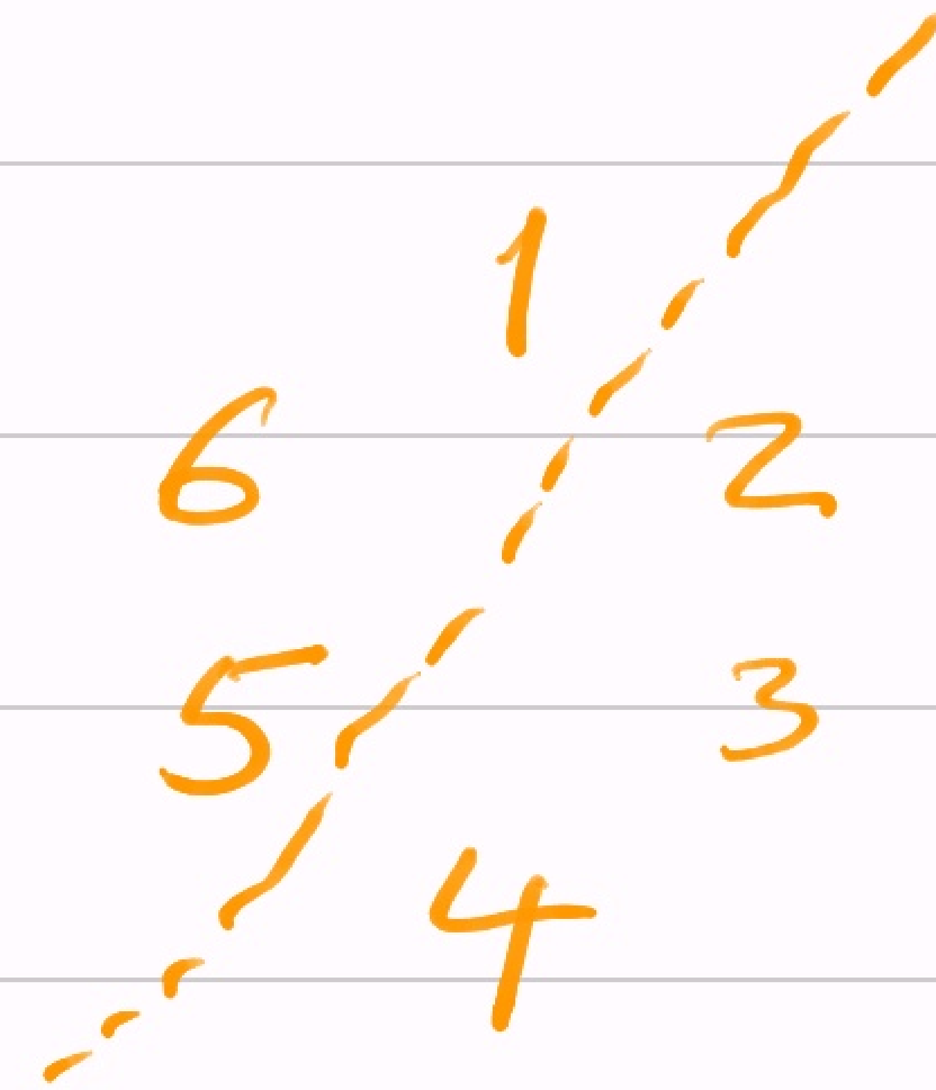
$$= \frac{1}{|G|}\cdot\sum_{g\in G}|A_g| = \frac{1}{2n}\cdot\sum_{g\in D_n} c^{n_g}.$$

- Let's do this for necklaces with $\underset{=n}{6}$ beads & $c=3$ colors:

— $G = D_6$ has 12 permutations:

$$\{e, r, \dots, r^5\} \sqcup \{s, sr, \dots, sr^5\}.$$

rotations   with reflections



- $n_e = n_{(1)(2)\cdots(6)} = 6.$
- $n_{r^t} = \gcd(t, 6)$
- $n_s = 3 = n = n_{sr^2} = n_{sr^4}$ $s = (12)(36)(45)$
- $n_{sr} = 4 = n_{sr^3} = n_{sr^5}.$ $sr = (13)(46)$

$$\Rightarrow \#\text{necklaces} = \frac{1}{12}\left(3^6 + 2\times 3^1 + 2\times 3^2 + 3^3 + 3\times 3^3 + 3\times 3^4\right)$$

$$= 1104/12 = 92 \ll 3^6.$$

- Qn: For $H \leq G$, is G/H again a group?
- eg. $2\mathbb{Z} \leq (\mathbb{Z}, +)$: $\mathbb{Z}/2\mathbb{Z}$ is the group $(\mathbb{Z}/2, +)$.

Normal subgroups.

- Defn: • $N \leq G$ is normal if $\forall g \in G$, $gN = Ng$ (i.e. left-coset = right-coset).

eg. $n\mathbb{Z} \leq (\mathbb{Z}, +)$ is a normal subgroup.

- $gNg^{-1} := \{ghg^{-1} \mid h \in N\}$ is called conjugate of subset $N \leq G$ by g.

▷ $N \leq G \iff gNg^{-1} \leq G, \ \forall g \in G$.

▷ normal $N \leq G \iff gNg^{-1} = N, \ \forall g \in G$.

- We can give G/H a group structure, if H is normal!

- $(gH) * (kH) := (gk)H$: Is $*$ well defined?

Theorem: Let $H \leq G$ be groups. The '$*$' is well defined on G/H iff H is normal.

Pf:

- Let $g_1, g_2 \in gH$ & $k_1, k_2 \in kH$.
- $g_1 H * k_1 H = g_1 k_1 \cdot H$ &

  $g_2 H * k_2 H = g_2 k_2 H$.
- But, $g_1 k_1 H = g_2 k_2 H$ iff $H = k_1^{-1} g_1^{-1} g_2 k_2 H$

  iff $k_1^{-1} \underbrace{g_1^{-1} g_2}_{=:h \in H} k_2 \in H$ iff $k_1^{-1} h k_2 \in H$

iff $kH \cdot k_2 \subseteq H$ (use: $\forall k_1, k_2 \in kH$) iff $kH = Hk$

iff H is normal (use: $\forall k \in G$).

Defn: Given normal $N \leq G$, the new group of cosets $(G/N, *)$ is called the quotient group or $G \bmod N$.

Corollary: • For abelian G, $\forall H \leq G$ give a group $G \bmod H$.

- e.g. $(\mathbb{Z}, +) \bmod n\mathbb{Z}$ is another way to see the group $(\mathbb{Z}/n, +)$.