Basic Counting

- Counting problems are basic problems in combinatorics & CSE. [eg. Probability]

- Two very simple rules of counting:

1) Sum rule: # ways to pick an element from two disjoint sets S_1 & S_2 $= |S_1| + |S_2|$.

2) Product rule: # ways to pick two elements from two sets S_1 & S_2 $= |S_1| \cdot |S_2|$.

- <u>Eg.1</u>: # numbers in $[1000, 9999]$ which are divisible by 3 $=$

  # 3-multiples in $[9999]$ $-$

  " " $[999]$ $= \frac{9999}{3} - \frac{999}{3}$.

- <u>Eg.2</u>: How many ways are there to put m balls into n bins $(m \geq n)$?

(1) Balls (resp. bins) are distinct: $= n^m$

(Prod.$\underset{\Longrightarrow}{\text{rule}}$) Pf: · Ball B_i has <u>exactly</u> n options, $\forall i \in [m]$.

$n \times - \times n = n^m$.

(2) Balls identical, bins distinct:

- number m $\longrightarrow$ partition into $(\leq n)$ Ordered parts.

$$\left[\begin{aligned} 4 &= 1+3 \\ &= 3+1 \\ &= 2+2 \\ &= 1+1+2 \end{aligned}\right]$$

- Think of $4=1+3$ as: $0|000\ ||$
  $= 2+2$ as: $00|00$

▷ $\# 0\text{'s} = m$ & $\# 1\text{'s} = (n-1)$.

Exercise: Balls/Bins $\rightsquigarrow$ ord. partition
$\rightsquigarrow$ 0/1 string.

- # permutations of this binary string $= \dfrac{(m+n-1)!}{m! \cdot (n-1)!}$

$$= \binom{m+n-1}{m} = \binom{m+n-1}{n-1}.$$

(3) Balls distinct, bins identical:

set $\{1, 2, \ldots, m\}$ partitioned into subsets/parts in an <u>unordered</u> way.

- We'll do this later under <u>Bell's number</u>.

(4) Balls (resp. bins) are identical:

Partition m into $(\leq n)$ <u>unordered</u> parts.

- We'll do this later under <u>generating function</u>.

— Counting is <u>not</u> unique. Sometimes counting in two ways unfolds interesting properties.

COUNTING in 2 WAYS

▷ In a conference people shook hands. #people who shook hands odd times, is even!

Pf:

- Consider $P := \{(p_i, p_j) \mid p_i \neq p_j \text{ shook hands}\}$.

  ▷ $|P|$ is even.

- $\#\{p_i \mid \#p_j\text{'s is odd}\} =: s_1$

  $\#\{p_i \mid$ " " even$\} =: s_0$

- Go over each $(p_i, p_j) \in P$ & check parity wrt p_i.

▷ $s_1 \cdot 1 + s_0 \cdot 0$ & $|P|$ have the same parity.

$\Longrightarrow$ s_1 & 0 " " " "

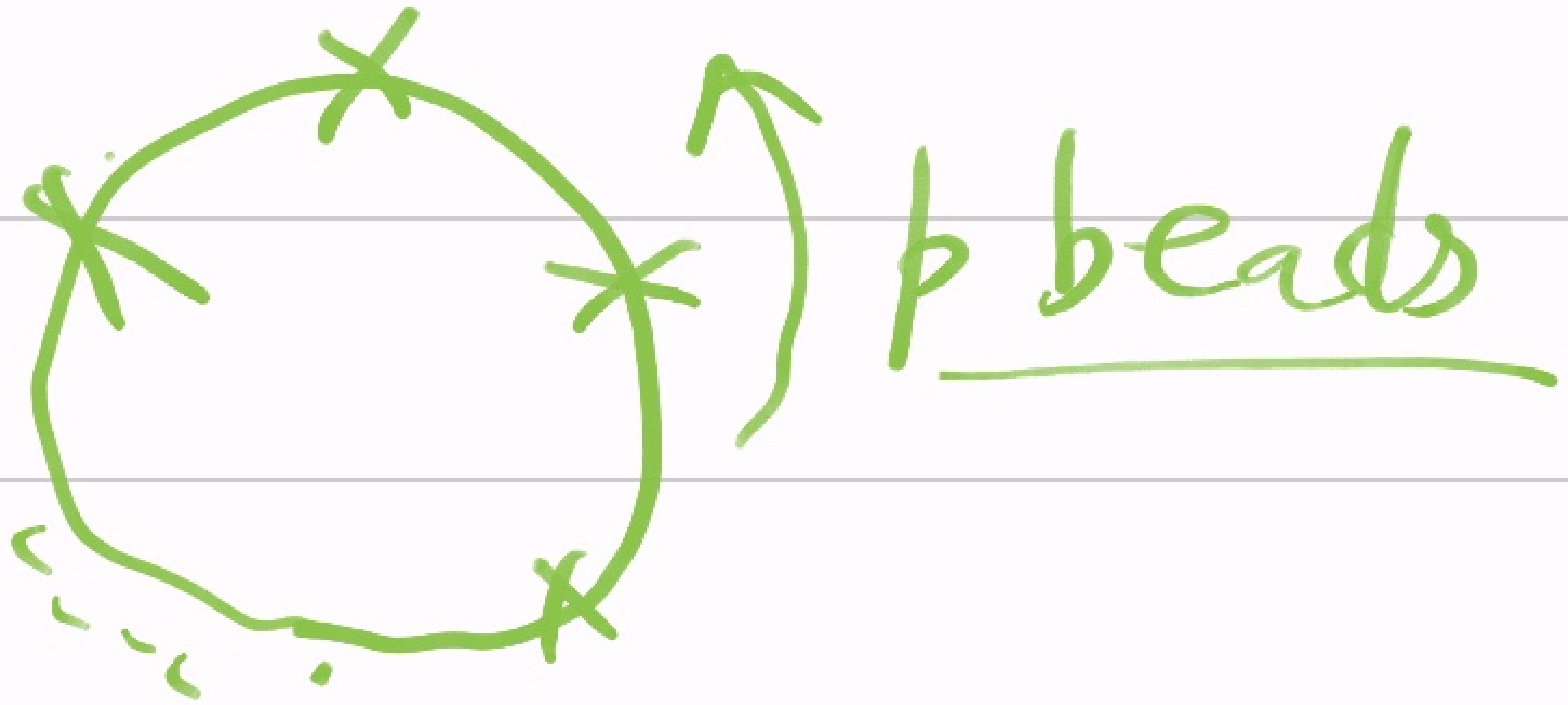
$\Longrightarrow$ $2 \mid s_1$. □

▷ Let $a \in \mathbb{N}$ & p be a prime. p divides $(a^p - a)$.

↑ necessary

Pf: • Consider a necklace with p beads.
• Each bead has a colors.

▷ # lines $= a^p$



Ex: p many line-necklaces correspond to the same (circle) necklace.

(Use: p prime ⇒ orbits are disjoint as long as it's not monochrome!)

$\Rightarrow$ #necklaces $= (a^p - a)/p + a$. □

Binomial coefficients

→ **Permutations:** Let S be a set of size n. # ways to pick (ordered) r elements

$$= n\cdot(n-1)\cdots(n-r+1) =: {}^{n}P_{r}.$$

▷ ${}^{n}P_{n} = n!$

→ **Combinations:** # ways to pick unordered r elements (or # r-subsets):

$$=: {}^{n}C_{r} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{r!\,(n-r)!}$$

$=: \binom{n}{r}$ (Binomial coeffs.)

▷ $(x+1)^n = \sum_{r} \binom{n}{r} x^r$. Pf: Exercise. $\square$

— Defn: $\binom{n}{-r} := 0$, for $r > 0$

$\binom{n}{r} := 0$, for $r > n$

$\binom{n}{0} := 1$.

▷ $\binom{n}{r} = \binom{n}{n-r}$ Pf: Bijection $r \longleftrightarrow n-r$ subsets. □

▷ $\sum_{r \in \mathbb{N}} \binom{n}{r} = 2^n$.

Pf: $\sum_{0 \le r \le n} \#(r\text{-subset}) = \#(\text{subsets}) = 2^n$. □

↳ Similar pf for Vandermonde's identity:

Theorem: $\displaystyle\sum_{0\le r\le k} \binom{n}{r}\cdot\binom{m}{k-r} = \binom{m+n}{k}$

(for given k, n, m)

Pf:

- $\binom{n}{r}\cdot\binom{m}{k-r} = \#(r\text{-subset of } [n])\ \#((k-r)\text{-subset of } \{n+1,\dots,n+m\})$

$\Rightarrow \displaystyle\sum_{r}$ " $= \#(k\text{-subset of } [m+n])$

$= \binom{m+n}{k}$. $\square$

→ Consider $(X+1)^n\cdot(X+1)^m = (X+1)^{n+m}$.

Ex: Reprove the Theorem.

Theorem (Pascal's identity): $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$.

Pf:
- Consider $x = 1$ in $S = [n]$.
- Any r-subset T $\begin{cases} \text{either} & x \in T \\ \text{or} & x \notin T. \end{cases}$
- By (invoking induction hypothesis) or defn of $\binom{\cdot}{\cdot}$:

$$\binom{n}{r} = \#(T \mid x \in T) + \#(T \mid x \notin T)$$

$$= \binom{n-1}{r-1} + \binom{n-1}{r}$$

$\square$

Qn: Use it in a recursive C-program?

$\approx 2^n$ steps! [**Ex:** Use dynamic programming.]

Recurrence relations

- Recursion is an aid to counting ; both in math. & CS.
- Eg. Fibonacci recurrence ?

- Pingala (~300 B.C.) wanted to count Shlokas.
- Shloka has short syllable (L = Laghu) → 1
  long " (G = Guru) → 2
- Q$_n$: # Shlokas with # beats = n ? ($= F_n$)
- Eg. $n=1$: L ; $n=2$: LL, G ; $n=3$: LLL, LG, GL, ..

Recurrence for F_n: $\begin{cases} \text{End with } L: & F_{n-1} \\ \text{or } \quad " \quad " \quad G: & F_{n-2} \end{cases}$

$\Rightarrow \quad F_n = F_{n-1} + F_{n-2}$. (Assume: $n \geq 2$)

$F_0 := 1 =: F_1$.

→ Rediscovered by Fibonacci (~1200)

→ How fast does F_n grow?

Generating functions

– To find a "closed form" expression of F_n.

→ Formal Power-Series: $G(t) := F_0 + F_1 \cdot t + F_2 \cdot t^2 + \ldots\ldots$

Think of t = very-small-real / t is a variable.

- Ex: Read about formal-power-series ring?

▷ Power-series can be added & multiplied.

→
$$\begin{array}{rccccc|c|ccl}
\cline{7-7}
G(t) & = & F_0 & + & F_1 t & + & F_2 t^2 & + & F_3 t^3 & + \cdots \\
t\cdot G(t) & = & & & F_0 t & + & F_1 t^2 & + & F_2 t^3 & + \cdots \\
t^2\cdot G(t) & = & & & & & F_0 t^2 & + & F_1 t^3 & + \cdots \\
\cline{7-7}
\end{array}$$

$\Rightarrow G(t)\cdot(1-t-t^2) = 1 + 0 \Rightarrow G(t) = 1/(1-t-t^2)$

- Say, $1-t-t^2 = (1-\alpha t)(1-\beta t)$; $\alpha, \beta := (1 \pm \sqrt{5})/2$

- Qn: Find $\mathrm{coef}(t^n)(G(t)) = ?$

- **Aim:** Write $G(t)$ as a sum of $\frac{1}{1-\alpha t}$ & $\frac{1}{1-\beta t}$

- Solve for: $\frac{1}{1-t-t^2} = \frac{c_1}{1-\alpha t} + \frac{c_2}{1-\beta t}$ $\left[\begin{array}{l} c_1 = \alpha/\sqrt{5} \\ c_2 = -\beta/\sqrt{5} \end{array}\right.$

$$= c_1 \cdot (1 + \alpha t + \alpha^2 t^2 + \dots) + c_2 \cdot (1 + \beta t + \beta^2 t^2 + \dots)$$

$\Rightarrow$ $F_n = coef(t^n)(G) = c_1 \cdot \alpha^n + c_2 \cdot \beta^n$

▷ $F_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{5}}$; $\forall n$.

→ This is a general way to solve linear recurr.

-Eg. $S_n = a_1 S_{n-1} + a_2 \cdot S_{n-2} + \dots + a_k \cdot S_{n-k}$.

for $a_1, a_2, \dots, a_k$ & k constants (independent of n)

- Let $G(t)$ be the generating fn. of S_n, $n \geq 0$.
- Like for F_n, we can deduce:

$$G(t) =: \frac{b_1 + b_2 t + \dots + b_k t^{k-1}}{1 - a_1 t - a_2 t^2 - \dots - a_k t^k} \overset{?}{=} \frac{c_1}{1-\alpha_1 t} + \dots + \frac{c_k}{1-\alpha_k t}$$

where, denominator $=: \prod_{i=1}^{k} (1-\alpha_i t)$.

Ex: This exists if α_i's distinct.

Ex: Over $\mathbb{C}$, $\exists\ \alpha_1, \dots, \alpha_k$ roots.

▷ Suppose denominator $= (1-\alpha t)^k$. Then, use:

$$\frac{1}{(1-\alpha t)^k} = (1-\alpha t)^{-k} = 1 + \binom{-k}{1}\cdot(-\alpha t) + \binom{-k}{2}\cdot(-\alpha t)^2$$

$$+ \cdots + \binom{-k}{n}\cdot(-\alpha t)^n + \cdots$$

where $\binom{-k}{n} := \dfrac{(-k)\cdot(-k-1)\cdots(-k-n+1)}{n!} = (-1)^n\cdot\binom{n+k-1}{n}$

$\Rightarrow \operatorname{coef}(t^n)(\cdot) = \binom{n+k-1}{n}\cdot\alpha^n = \binom{n+k-1}{k-1}\cdot\alpha^n$

Theorem: Linear recurrence for $S_n \Longrightarrow$
$S_n =$ linear combination of functions
e.g. α_i^n, $n^j\cdot\alpha_i^n$

Ex: Complete the proof.

- eg. S_n could be $1+ \quad 2^n + \quad 3^n + \quad n \cdot 3^n$.

$\Rightarrow$ Show that $S_n = a_1 S_{n-1} + a_2 S_{n-2} + a_3 S_{n-3} + a_4 S_{n-4}$

- What happens if $G(t)$ is <u>not</u> rational?

$\rightarrow$ eg. $G(t) = e^t$ or e^{e^t}?

- eg. $S_n = n \cdot S_{n-1} \Rightarrow S_n = n!$

$\Rightarrow G(t) = 1 + 1!t + 2!\,t^2 + 3!\,t^3 + \dots$

Exponential Generating Function (egf)

- In some cases, when S_n's recurrence is non-linear, we can analyse using egf.

- Eg. *Involution* is a permutation σ on $[n]$ s.t. $\forall i \in [n]$, either $\sigma(i)=i$ or $\exists j$, σ swaps i,j.

- eg. $\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ll, bend left] \end{tikzcd}$ is *not* an involution.

- eg. $\begin{tikzcd} 1 \arrow[r] & 2 \arrow[l, bend left] \end{tikzcd}$ & $3 \to 3$ is ″ ″.

- Let $I(n) := \#(\text{involutions on } [n]) \leq n!$.

- eg. $I(0) := 1$, $I(1) = 1$, $I(2) = 2$, $I(3) = 4$.

$\{id, (12), (23), (31)\}$

- Recurrence for $I(n)$: Consider element n.

(1) n is fixed : # involutions $= I(n-1)$

(2) n swaps with j: " $= I(n-2)$

#j's $= (n-1)$

$$\Rightarrow \quad I(n) = I(n-1) + \underbrace{(n-1)}_{\to \text{non-constant}} \cdot I(n-2).$$

Ex: $\forall n \geq 2$, $I(n)$ is even. $I(n) > \sqrt{n!}$.

↳ Use induction.

- Consider $\theta(t) := \sum_{n\geqslant 0} \frac{I(n)}{n!} \cdot t^n$

eg. $\forall n, I(n) = 1 \implies \theta(t) = e^t$. (hence, egf!)

- Use derivative: $\frac{d\theta}{dt} = \sum_{n\geqslant 1} I(n) \cdot \frac{t^{n-1}}{(n-1)!}$

$$= \sum_{n\geqslant 1} \left(I(n-1) + (n-1)\cdot I(n-2)\right) \cdot \frac{t^{n-1}}{(n-1)!}$$

$$= \sum_{n\geqslant 1} I(n-1) \cdot \frac{t^{n-1}}{(n-1)!} + t \cdot \sum_{n\geqslant 2} I(n-2) \cdot \frac{t^{n-2}}{(n-2)!}$$

$$= \theta(t) + t \cdot \theta(t).$$

$$\Rightarrow \quad \frac{d(\log\theta)}{dt} = 1+t$$

$$\Rightarrow \quad \log\theta = t+\frac{t^2}{2}+c$$

$$\Rightarrow \quad \theta(t) = e^{t+t^2/2+c} .$$

$$\Rightarrow 1 = I(0) = e^c \quad \Rightarrow \quad c=0$$

$$\Rightarrow \quad \theta(t) = e^{t+\frac{t^2}{2}} .$$

▷ Use this to compute $\frac{I(n)}{n!} = \mathrm{coef}(t^n)\left(e^{t+\frac{t^2}{2}}\right).$

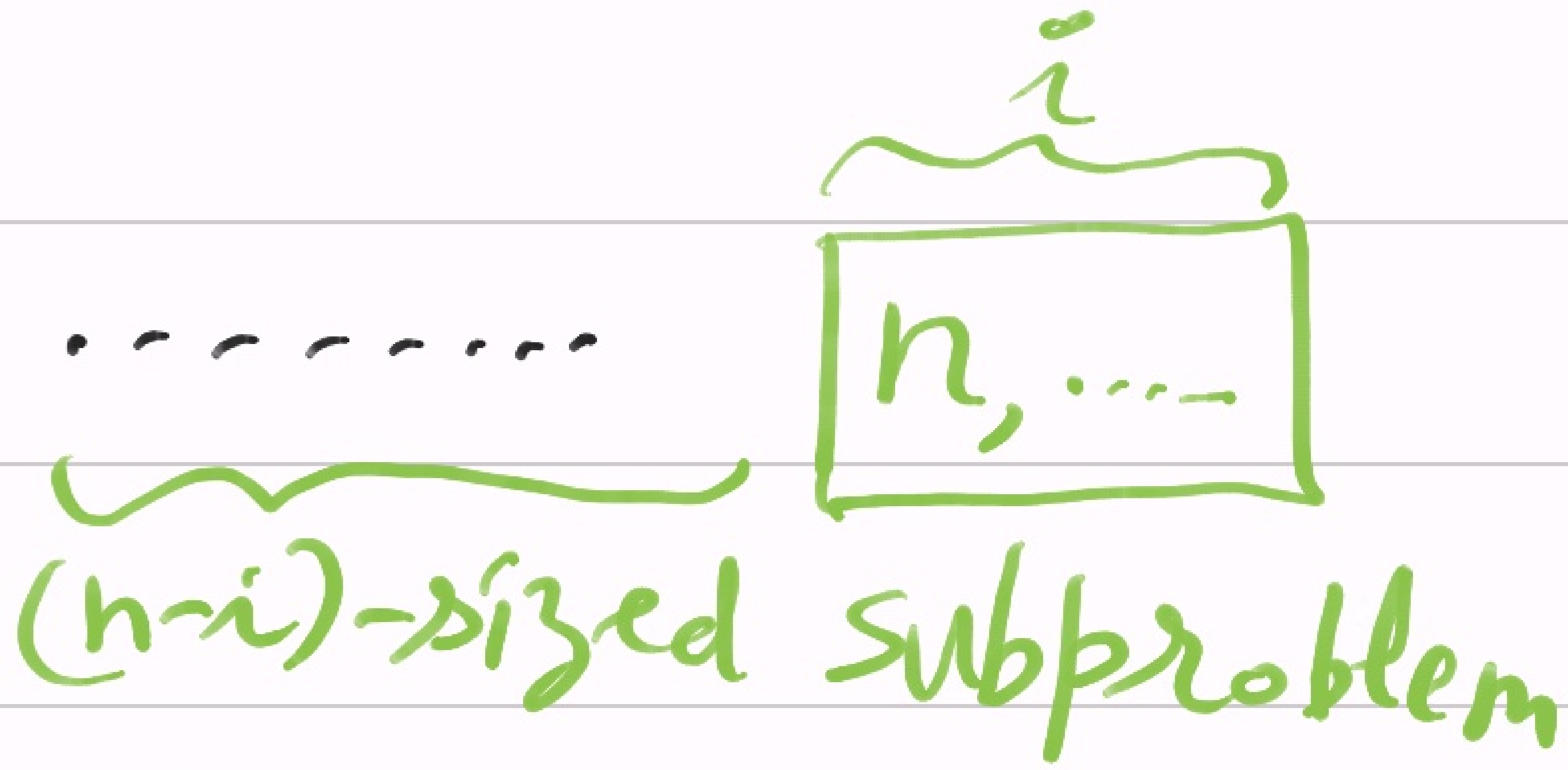
- Another eg. of egf:

#partitions of the set $[n]$

= # put n distinct balls into n identical bins

=: B_n

▷ $B_n = \sum_{i\in[n]} \binom{n-1}{i-1} \cdot B_{n-i}$



Pf: • In a partition, consider the part which has n. Let it be of size =: i.

- # such parts/bins = $\binom{n-1}{i-1}$
- # partitions on (n-i) element =: B_{n-i}.

By sum/prod-rules: $\sum_i \binom{n-1}{i-1} \cdot B_{n-i} = B_n$. □

- Define $\underline{\theta(t)} := \sum_{k\geq 0} B_k \cdot t^k/k!$ [egf of B_n]

- Study $\frac{d\theta}{dt}$. Use Recurrence to get

$$(d\theta/dt) = \theta \cdot e^t$$ (Exercise)

$\Rightarrow$ <u>Exercise:</u> $\theta(t) = e^{e^t-1}$.

B_n is called <u>Bell's</u> number.