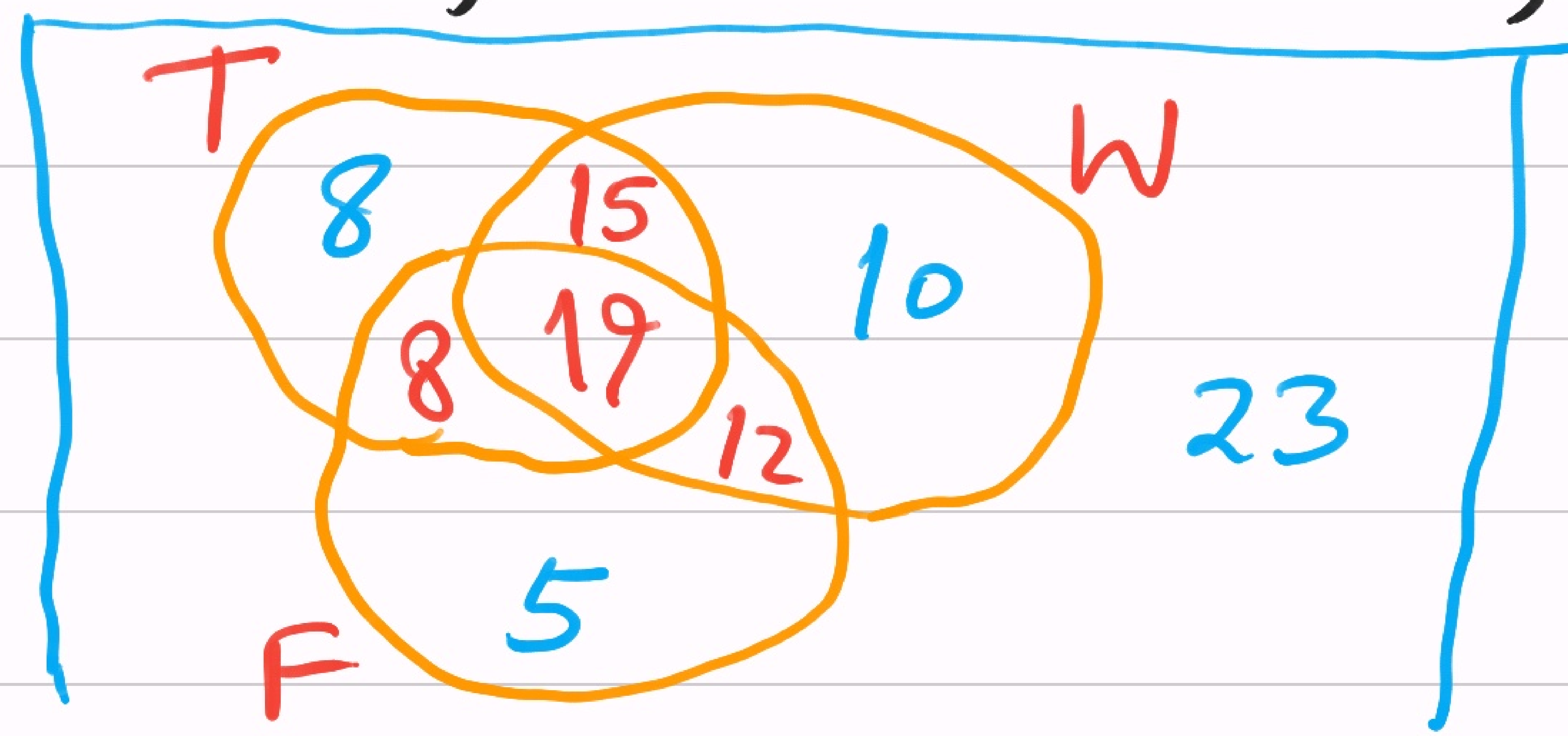
# Some Combinatorial techniques

## Inclusion-Exclusion:

- eg. 100 students. They use W, F, T.
  - 44 use F ; 50 use T ; 56 use W.
  - 27 use $F \cap T$ ; 31 use $F \cap W$ ; 34 use $W \cap T$ ;
  - 19 use $F \cap T \cap W$.

$\to$ We filled this Venn diagram by moving inside to outside.



- Let us try to do this <u>outside</u> to <u>inside</u>? This will help when the #sets is <u>large</u>.

- Let $U :=$ set of students. $A_1, \dots, A_n$ be subsets of $U$.

→ For every subset $I \subseteq [n]$, $\underline{A_I} := \bigcap_{i \in I} A_i$.

- We're interested in $\left(\bigcup_{i\in[n]} A_i\right)^c := U - \bigcup_{i\in[n]} A_i$.

<u>Claim</u>: $\left|\left(\bigcup_{i\in[n]} A_i\right)^c\right| = \sum_{I\subseteq[n]} (-1)^{|I|} \cdot |A_I|$. $(A_\phi := U.)$

[Principle of Incl.-Excl.]

Proof: • Idea – Fix $u \in U$ & check how's it being counted in RHS.

• Case 1: $u \in \left(\bigcup_{i \in [n]} A_i\right)^c \Rightarrow u \notin A_i, \forall i.$

& $u \in U = A_\phi$.

$\Rightarrow$ $u$ is not counted in any $|A_I|$ except $|A_\phi|$.

$\Rightarrow$ RHS count for $u$ is $1$

& LHS " " " " 1. Done!

• Case 2: $u \notin \left(\bigcup_{i \in [n]} A_i\right)^c \Rightarrow u \in A_i$, for some $i$.

• Consider $J_u := \{ j \in [n] \mid u \in A_j \}$. (non-empty)

$|J_u| =: k > 0$.

$\triangleright\ u \in A_I \iff I \subseteq J_u$.

$[\,|J_u| = k\,]$

$\Rightarrow \text{RHS} = \sum_{I \subseteq J_u} (-1)^{|I|} \cdot 1$

• For $|I| = \ell$, the contribution above is:

$$(-1)^{\ell} \cdot \binom{k}{\ell}$$

$$\Rightarrow \text{RHS} = \sum_{\ell=0}^{k} (-1)^{\ell} \cdot \binom{k}{\ell} = (1-1)^k = 0.$$

$\Rightarrow$ $\forall u \in U$, the count on RHS is correct.

$\square$

- Eg. of Derangements: Put $n$ distinct letters into $n$ addressed envelopes. Every letter goes into a wrong envelope.
- Frame this as $\left(\bigcup_{i\in[n]} A_i\right)^c$ ?

- Define $A_i :=$ arrangements with letter-$i$ correct.
  $\Rightarrow \bigcup_{i\in[n]} A_i =$ some letter correct.

- [PIE] $\Rightarrow$ #derangements $= \sum_{I\subseteq[n]} (-1)^{|I|} \cdot |A_I|$

▷ $|A_I| = (n-|I|)!$

$\Rightarrow$ #derang. $= \sum_I (-1)^{|I|} \cdot (n-|I|)! =$

$$= \sum_{\ell=0}^{n} (-1)^{\ell} \cdot (n-\ell)! \cdot \binom{n}{\ell}$$

$$= n! \cdot \sum_{\ell=0}^{n} (-1)^{\ell} / \ell! \approx \frac{n!}{e}$$

# Pigeonhole Principle (php)

**Theorem 1:** If there are $n+1$ pigeons & $n$ pigeonholes then $\exists$ pigeonhole with $>1$ pigeons.

Pf:
- Let $a_1 \geq \cdots \geq a_n$ be the sizes of the pigeonholes.
- $$a_1 \geq \frac{a_1 + \cdots + a_n}{n} = \frac{n+1}{n} > 1$$
- Since $a_1 \in \mathbb{N} \Rightarrow a_1 \geq 2$. $\square$

(Averaging principle)

- Eg.1. There are 367 people in a group.
$\Rightarrow \exists \geq 2$ people with same birthday.
[ php: People $\to$ Pigeons ; Days $\to$ Pigeonholes.]

- Eg.2. Facebook has $n$ users. $\exists \geq 2$ people who have the same #friends.
$\{u_1, u_2, \dots, u_n\} \longrightarrow [0, 1, \dots, n-1]$

Pf: Case 1: $\exists u_i$ with #friends $= 0$.
Let's $i=1$ (wlog). #friends$(u_1) = 0$.
$\{u_2, \dots, u_n\} \longrightarrow \{1, 2, \dots, n-2\}$
$\Rightarrow$ (by php) $\exists u_i \neq u_j$, $\#(u_i) = \#(u_j) \Rightarrow$ done!

If $\exists i > 1$, $\#(u_i) = 0$ $\Rightarrow$ done!

• Case 2: $\forall u_i$, $\#friends(u_i) \geq 1$.

$\{u_1, \to u_n\} \to \{1, 2, - \to n-1\}$

$\Rightarrow$ (by php) $\exists u_i \neq u_j$, $\#fr(u_i) = \#fr(u_j)$.

$\Rightarrow$ done! $\square$

Eg 3.: Given $n \in \mathbb{N}_{>0}$, $\exists m \in \mathbb{N}$ containing only 0/1 digits & divisible by $n$. $(n|m)$

Pf: • Consider the pigeons as unary numbers:

$\{1, 11, 111, 1111, \ldots\}$

• Pigeonholes are $\{0, 1, \to n-1\}$. $(u_i \% n)$

$\Rightarrow$ (by php) $\exists r$: $u_1$ & $u_2$ have remainder $r$ mod $n$.

$\Rightarrow$ (Say, $u_1 > u_2$) $\quad u_1 - u_2^{\neq 0}$ is $1^* 0^*$ &

$n \mid u_1 - u_2 =: m$. $\quad \square$

Eg 4. Let $x$ be an irrational real number ($x \in \mathbb{R} \setminus \mathbb{Q}$). Then, $\forall n \in \mathbb{N}_{>0}$, $\exists\, p, q \in \mathbb{Z}$ s.t.

$$\left| x - \frac{p}{q} \right| < \frac{1}{nq} \quad \& \quad q \in [n].$$

[Rational approximation of $x$]

[Converse holds true!]

**Theorem 2** (averaging): $rn+1$ pigeons & $n$ pigeonholes $\Longrightarrow$ $\exists$ pigeonhole with $> r$ pigeons.

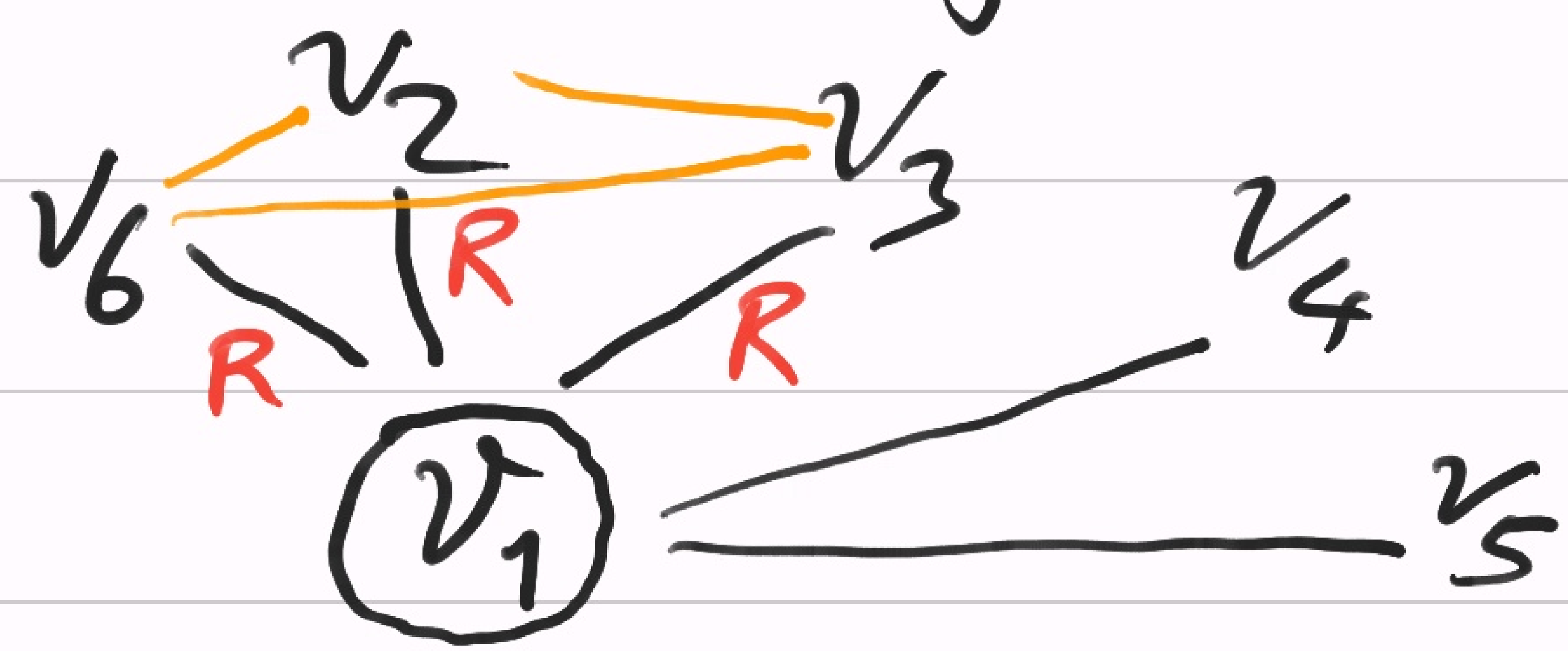
Pf: • $a_1 \geq a_2 \geq \cdots \geq a_n$ be the sizes.

• $a_1 \geq \dfrac{a_1 + \cdots + a_n}{n} = \dfrac{rn+1}{n} > r$. $\square$

Eg. 5: In a hexagon there are $\binom{6}{2}$ lines; colored R/B. $\Longrightarrow$ $\exists$ monochromatic triangle.

Pf: • Focus on $v_1$.

• $\exists$ 3 edges of the same color, say R. (Thm 2).



- Consider the triangle $v_2 v_3 v_6$.

↳ Either it's Blue colored $\Rightarrow$ done!

↳ Or, there's a Red edge

$\Longrightarrow$ We get a Red-$\Delta$ with $v_1$. $\square$

<u>Exercise</u>: Show that the proof is tight; i.e. it's not true for pentagon.

# Combinatorics using linear algebra

- In many interesting combinatorial constructions algebra is a major tool.

[ Babai & Frankl — Lr. algebra methods in Combinatorics ]

- Eg. "Clubs with members in a College" (Maximize)
  - There are $n$ people in the college.
  - Each club size is even.
  - Intersection of clubs is even.

$\triangleright$ $2^n > \#clubs > 2^{n/2}$. [Pf: Pair up the people. $\square$]

- Let's now change the conditions to:
  - each club size is <u>odd</u>, &
  - intersection of clubs is <u>even</u>.

<u>Qn:</u> #clubs now = ? <u>$n$</u>?

- <u>Idea:</u> • Ambient vector space := $\mathbb{Q}^n$ or $\mathbb{R}^n$.
  • A club is a <u>point</u>/<u>vector</u> in this space.

- Eg. club = $\{1, 3, 10\} \longrightarrow (1,0,1,0,0,0,0,0,0,1, \ldots\ldots)$

- Let there be $m$ clubs : $v_1, v_2, \ldots, v_m \in \mathbb{Q}^n$.
- $\forall i \in [m]$, $v_i^T \cdot v_i$ is odd. (column-vector)
- $\forall i \neq j \in [m]$, $v_i^T \cdot v_j$ = size-of-intersection = even.

**Claim:** $v_1, \ldots, v_m$ are $\mathbb{Q}$-linearly-independent. ($\Rightarrow m \leq n$)

Pf: • Assume not. $\exists\, \alpha_1, \ldots, \alpha_m \in \mathbb{Z}$ s.t.

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$, & $\exists i, \alpha_i \neq 0$.

$$\Rightarrow \alpha_1 \underbrace{v_i^T v_1}_{\text{even}} + \alpha_2 \underbrace{v_i^T v_2}_{\text{even}} + \cdots + \underset{\text{even?}}{\alpha_i} \underbrace{v_i^T v_i}_{\text{odd}} + \cdots + \alpha_m \underbrace{v_i^T v_m}_{\text{even}} = 0.$$

Idea: Consider $\{i \in [m] \mid \alpha_i \neq 0\} \Rightarrow$ W.l.o.g. $\alpha_1 = \text{odd}$.

$$\Rightarrow \quad \underbrace{\alpha_1}_{\text{odd}} \cdot \underbrace{v_1^T v_1}_{\text{odd}} + \underbrace{\alpha_2}_{\in \mathbb{Z}} \cdot \underbrace{v_1^T v_2}_{\text{even}} + \dots + \underbrace{\alpha_m}_{\in \mathbb{Z}} \cdot \underbrace{v_1^T v_m}_{\text{even}} = 0$$

$\Rightarrow \lightning \Rightarrow \{v_1, \dots, v_m\}$ are $\mathbb{Q}$-l.i.

$\Rightarrow m \le n.$ $\square$

- Eg. $\{\{1\}, \{2\}, \dots, \{n\}\}$ are <u>n clubs</u>.