▷ S is a vertex cover $\iff$
$V-S$ is an independent set. [Pf: $\bar{S}$ has no edge. □]

▷ Thus, least vertex cover &
maximum indep. set & maximum clique are
all equivalent problems!

Qn: Approximate indep. set is hard?

Coloring –

– $(s_1, s_2) \in E$ if s_1, s_2 are friends.
– Use exam. hall. nos. as colors on the vertices.

- <u>Defn</u>: • Given $G=(V,E)$, a <u>coloring</u> is a map from $c: V \to$ colors, s.t. $(u,v) \in E \Rightarrow c(u) \neq c(v)$.

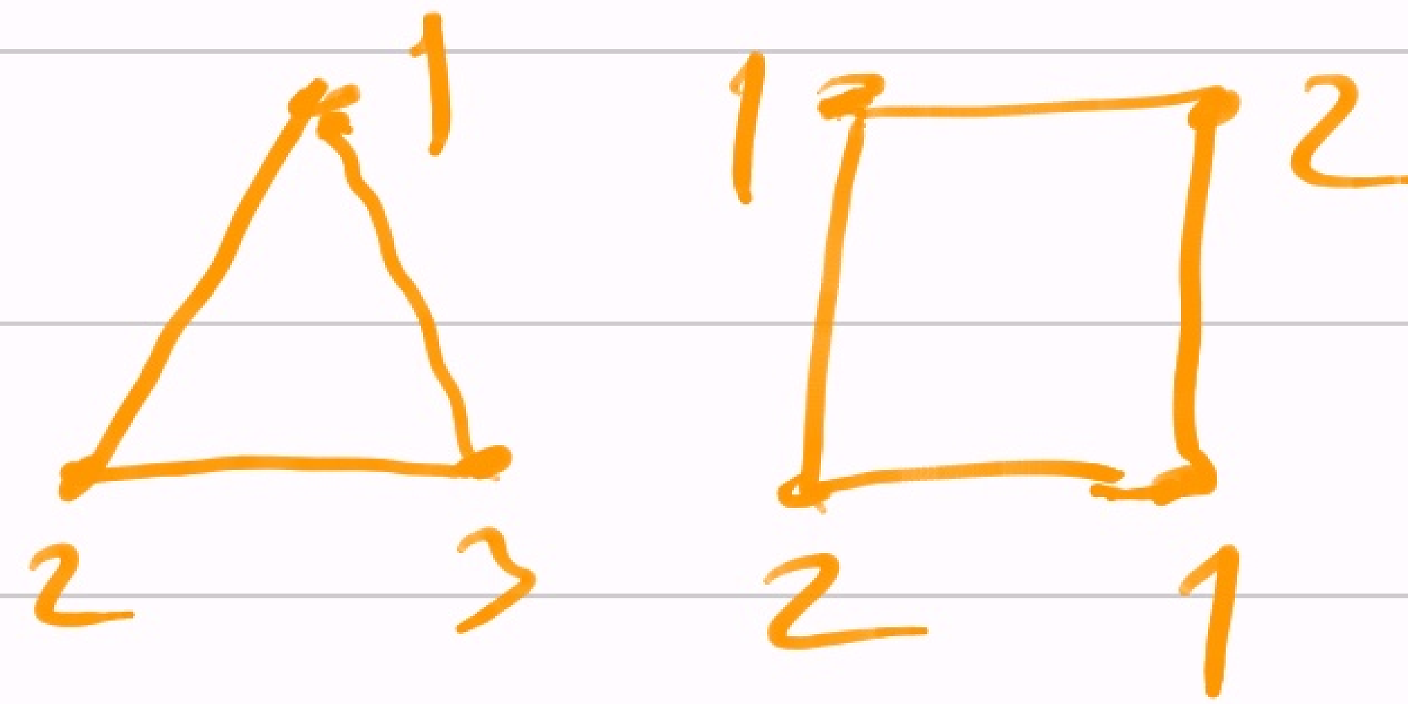
• Minimum <u>number</u> of colors needed for G is called <u>chromatic number</u> <u>$\chi(G)$</u>.

- Eg.1: Map of countries/states $\Rightarrow \chi(G)=4$ [Appel-Haken '76]

- Eg.2: Social graph $G \Rightarrow \chi(G)=$ min. number of examination halls needed.

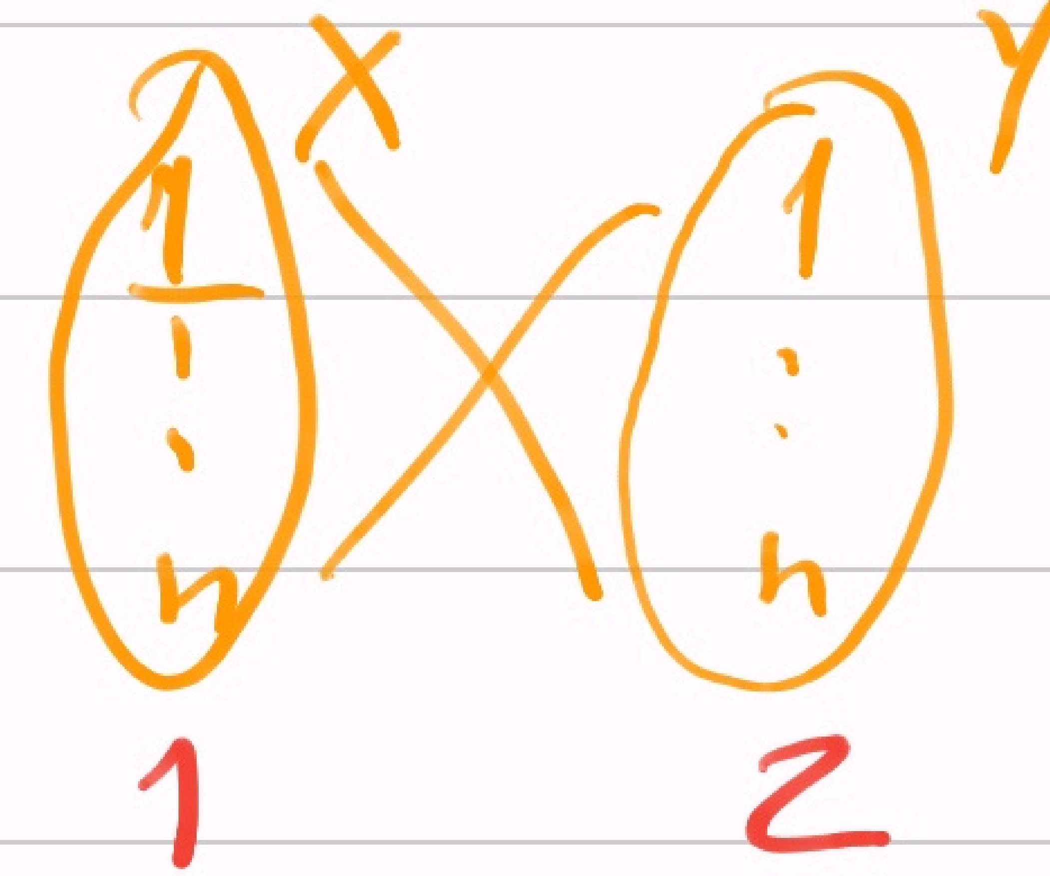
<u>Qn</u>: Compute $\chi(G)$, given input G, in a <u>fast</u> way?

- Eg. $\chi(C_n) = \begin{cases} 3, & n \text{ odd} \\ 2, & n \text{ even} \end{cases}$



- Eg. $\chi(K_{n,n}) = 2$

  $\chi(\text{bipartite}) \le 2$.



▷ $\chi^{-1}(1)$ is independent set of G.

▷ $\chi(G) \le k \iff G$ can be partitioned into $\le k$ disjoint IS.

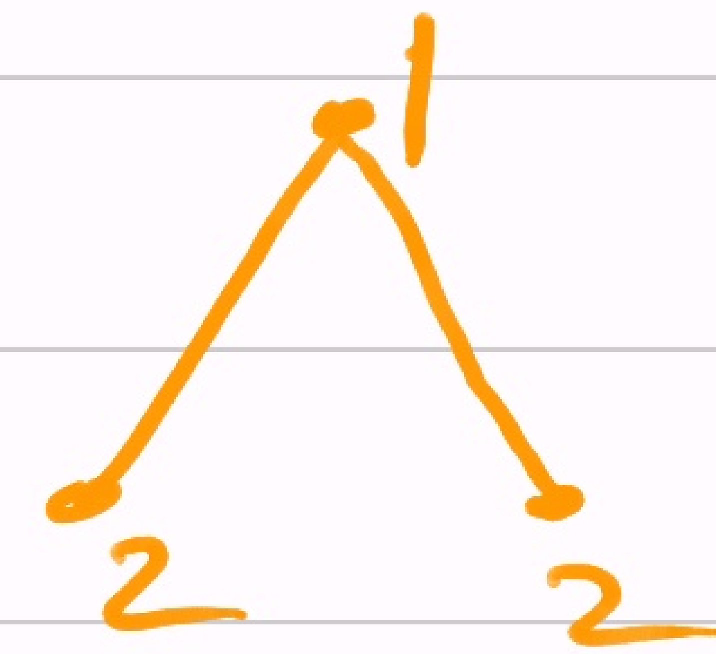
<u>Theorem:</u> $\alpha(G) \cdot \chi(G) \ge |V| =: n$.

<u>Pf:</u> · Let the optimal coloring of G <u>partition</u> $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ $(k := \chi(G))$.

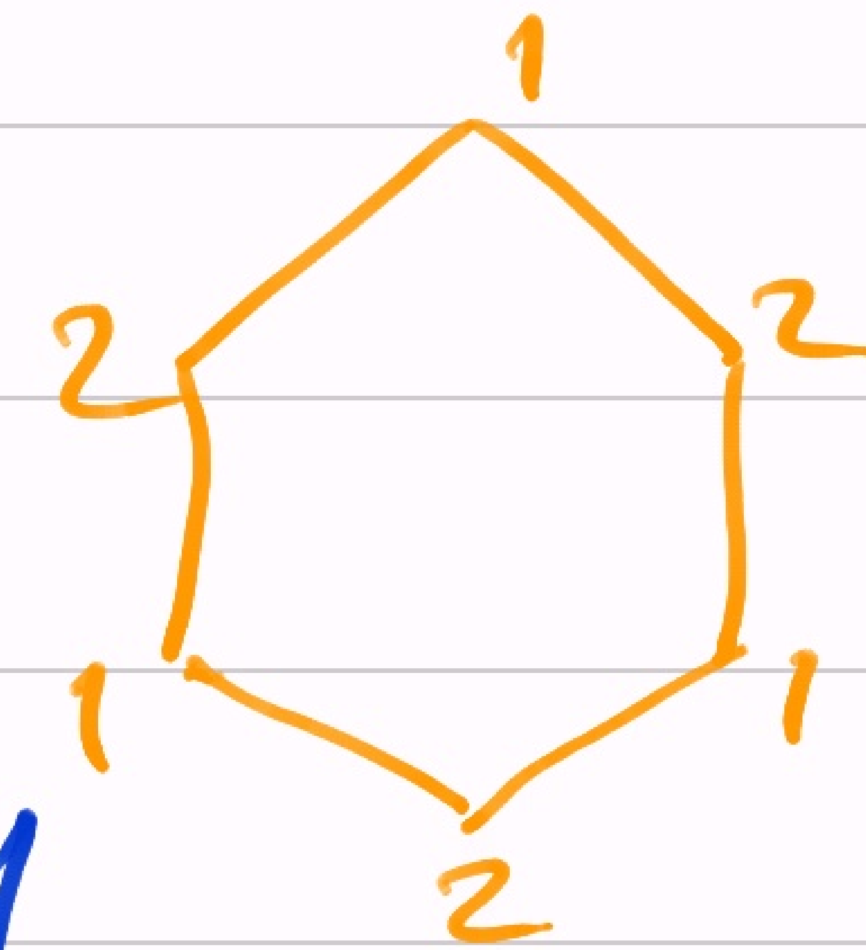
$$\Rightarrow n = |V| = \sum_{i=1}^{k} |V_i| \le \sum_{i=1}^{k} \alpha(G) = k \cdot \alpha(G)$$

$$\Rightarrow \quad n \le \chi(G) \cdot \alpha(G). \qquad \square$$

- Eg.1: $\alpha(G) = 2$, $\chi(G) = 2$, $n = 3$;



- Eg.2: $\alpha(K_n) = 1$, $\chi(K_n) = n$



Theorem: Max. deg in G is $k \Rightarrow \chi(G) \le k+1$.

If G is connected $\exists v$, $\deg(v) < k \Rightarrow \chi(G) \le k$.

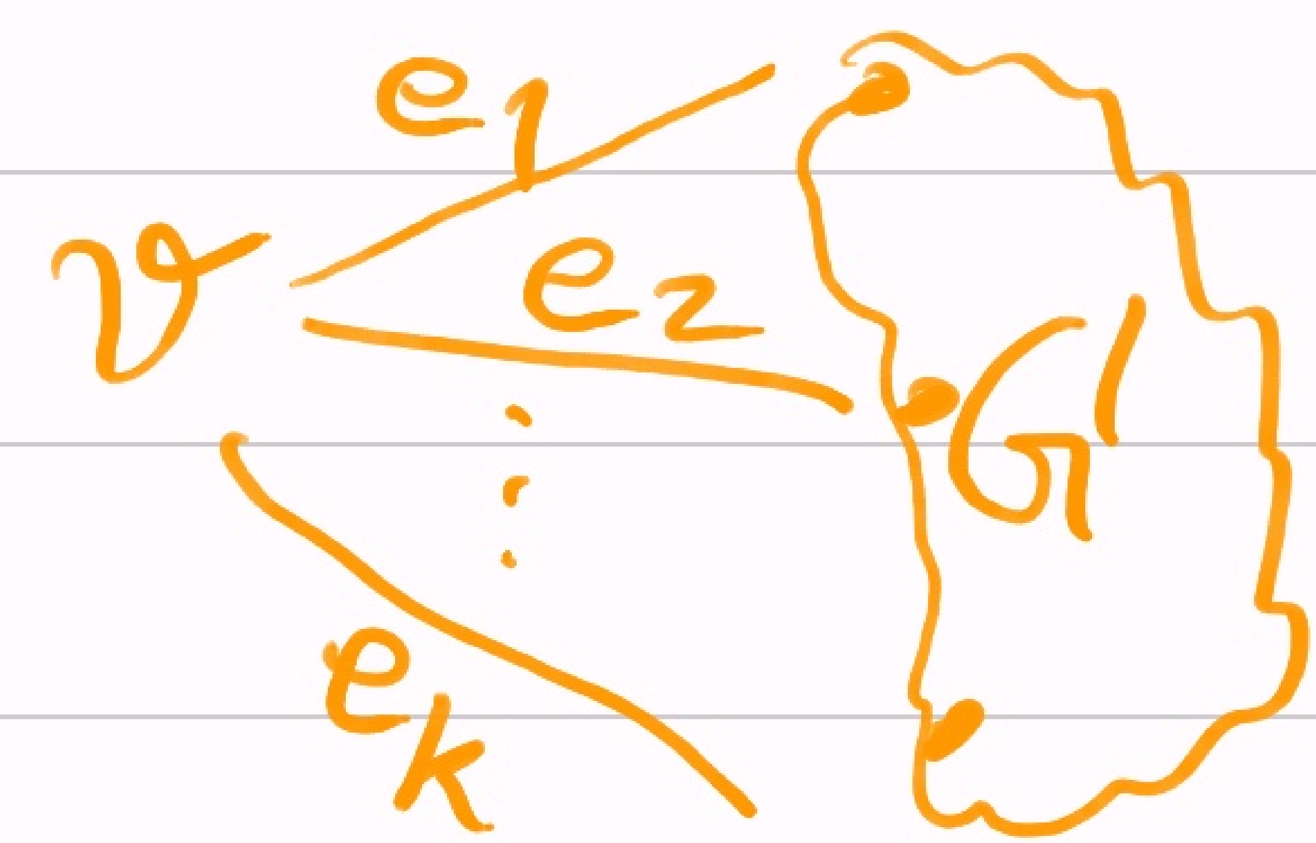
Pf: (1) Apply <u>induction</u> on $|V(G)| =: n$.

- <u>Base:</u> $n \leq k+1$.
- Let $v \in V$ be any vertex. Consider G' by deleting v from G. $= (V\setminus\{v\}, E\setminus N(v))$.

$\Rightarrow$ (induction hyp.) $\chi(G') \leq k+1$.

$\Rightarrow$ neighbors $N(v)$ use up at most k colors.

$\Rightarrow$ v has ≥ 1 color available

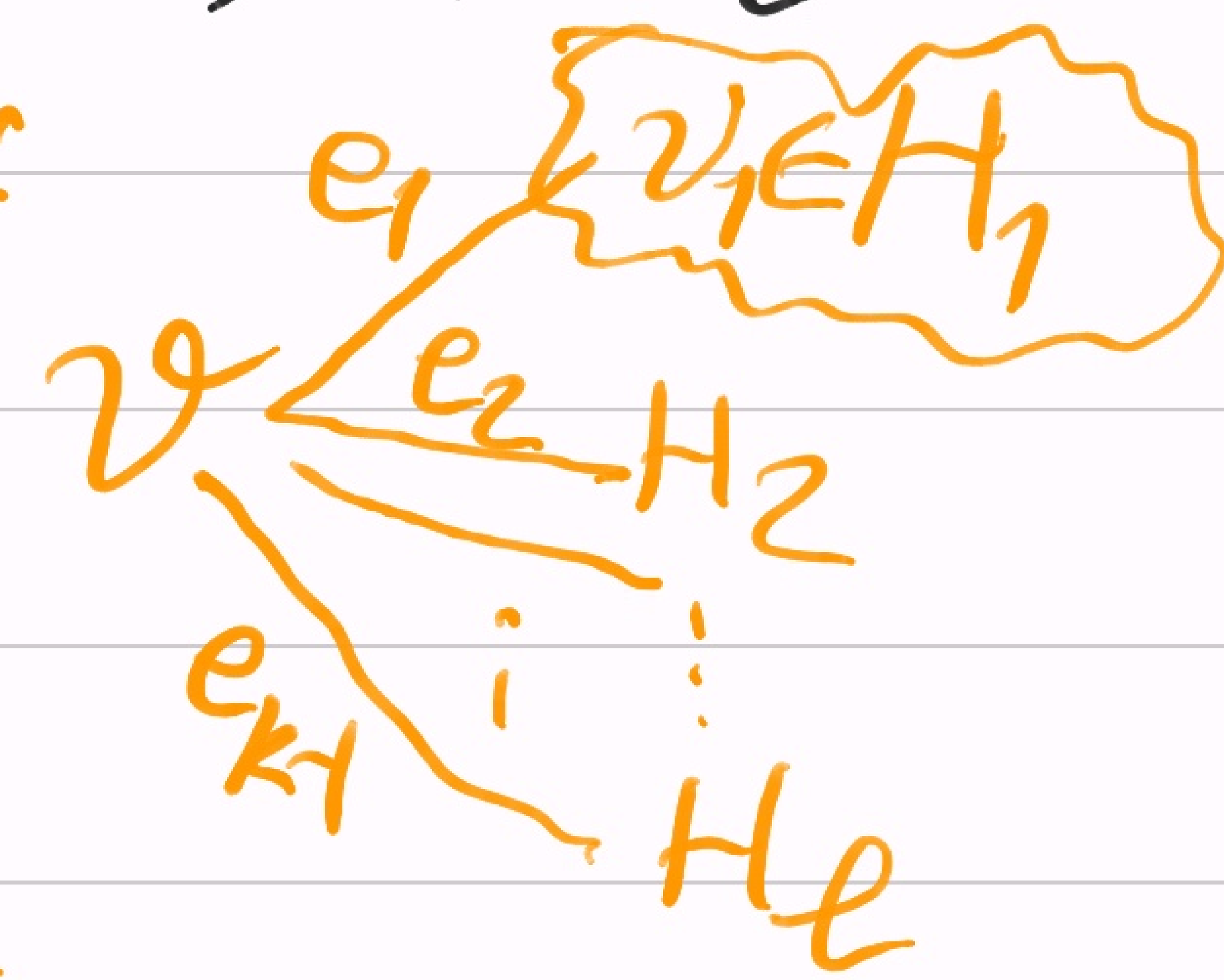


$\Rightarrow$ $\chi(G') \leq k+1$.

(2) We'll use induction on n, and improve using $\deg(v) < k$.

• $G' := G \setminus \{v\}$. Let $H_1, H_2, \to H_\ell$ be its connected components.

G:



▷ H_1 is connected & $\exists v_1, \deg(v_1) < k$.

$\Rightarrow$ (ind. hyp. on H_1) $\chi(H_1) \leq k$.

" — $\Rightarrow$ $\chi(H_i) \leq k, \forall i$.

$\Rightarrow$ neighbors $N(v)$ use up $k-1$ colors.

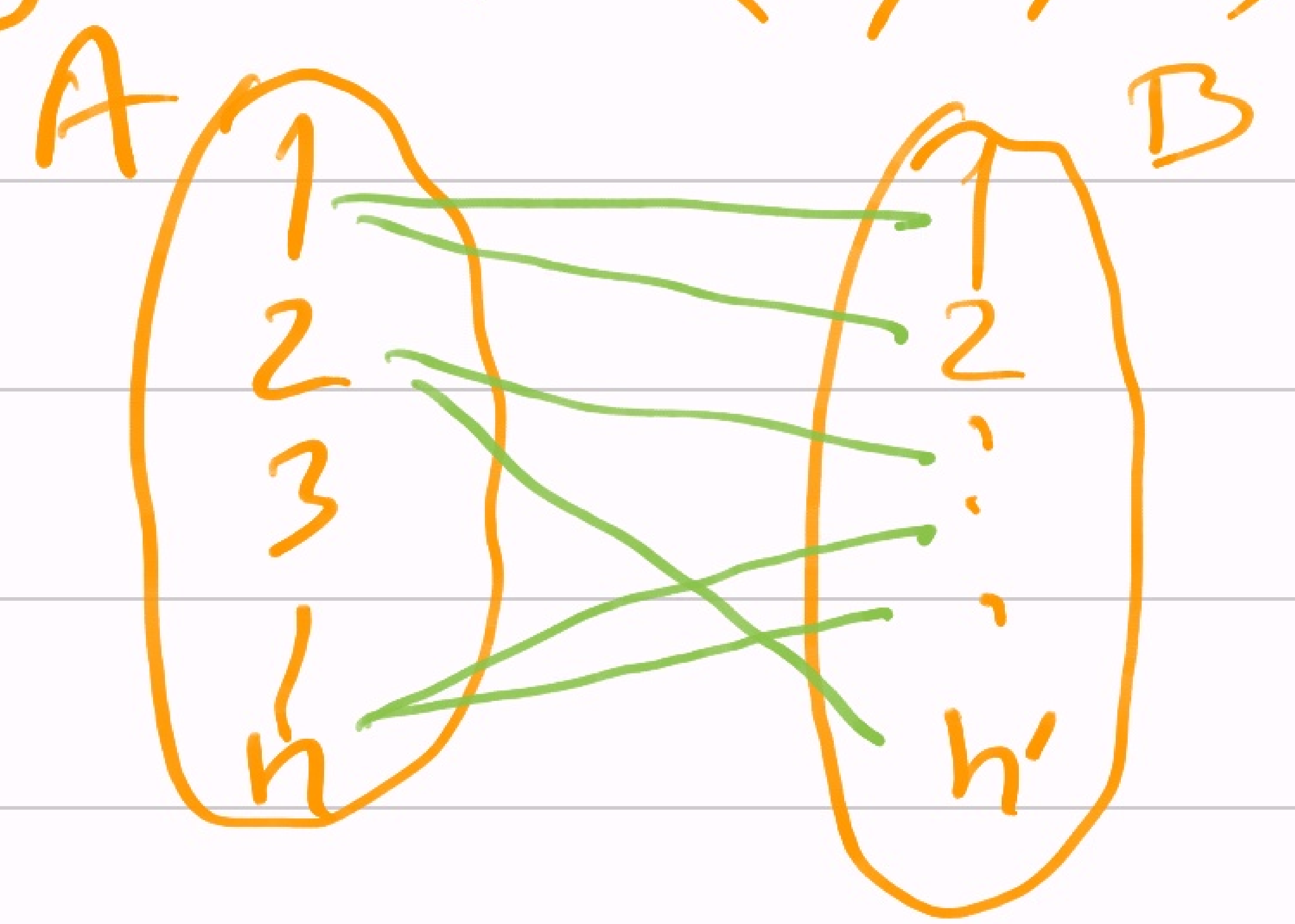
$\Rightarrow$ v has ≥ 1 color available.

$\Rightarrow \chi(G) \leq k$. □

— Eg. χ(star on n vertices) $= 2$; $\deg = n-1$

Matching

- Eg. Tennis tournament between teams A & B. Given graph of player pairings: $G=(A,B,E)$
- Qn: Max. # matches that can be played simultaneously?

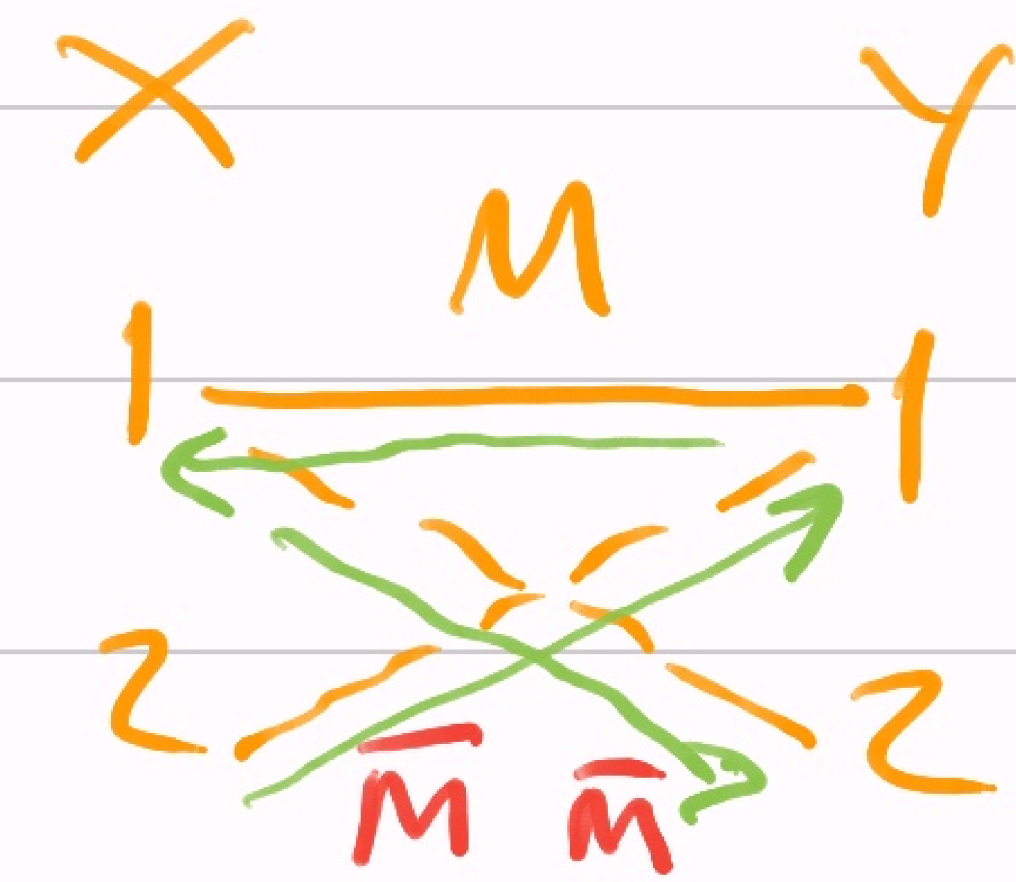


- I.e. find max. # non-overlapping edges in $E(G)$.
- Defn: • Matching M in bipartite $G =: (X \sqcup Y, E)$ is a subset of edges s.t. no two edges share a vertex.

- <u>Maximum</u> matching M is the largest possible in #edges.
- <u>Maximal</u> matching M is a matching s.t. $\forall e \in E \setminus M$, $M \sqcup \{e\}$ is <u>not</u> matching.

Eg. $E(G) = \{11, 12, 21\}$; $M := \{11\}$ is maximal, but $M' := \{12, 21\}$ is maximum & larger!

G:



<u>Qn:</u> 1) Algo. for maximal matching? (easy)

2) " " maximum " ? (nontrivial)

Idea: • What if we consider the alternating path $P: 2 \underset{\overline{M}}{\rightarrow} 1 \xrightarrow{M} 1 \underset{\overline{M}}{\rightarrow} 2$ with $\overline{M}$-edges one more than M-edges.

(symmetric difference)

• Now delete the M-edges: $P \triangle M := (P \setminus M) \sqcup (M \setminus P)$.

- Defn: • Given matching M in $G = (X \sqcup Y, E)$; an alternating path P is s.t.

(i) vertex-1 in P is in X and last-vertex is in Y, and both are unmatched (i.e. no M-edge is incident)

(ii) edges alternate between $\overline{M}$ & M.

Theorem: Matching M is maximum in G <u>iff</u> $\nexists$ alternating path in G.

Pf: $(\Rightarrow)$: Suppose $\exists$ alternating path P in G.

Consider $M' := P \Delta M := (P \setminus M) \sqcup (M \setminus P)$.

$\triangleright$ M' is a matching & one larger than M.

P: 

$(\Leftarrow)$: Suppose M is not maximum.

Let M' be some matching in G, that's maximum.

- Let G' be the subgraph of G with edges only from M & M'. $\triangleright$ $\deg(G') \leq 2$. So, connected components in G' are $\{$vertex, path, cycle$\}$.

$\Rightarrow$ any cycle in G' is even & with alternating M-edges & M'-edges.

$\Rightarrow$ $\triangleright$ $\exists$ connected-component(G') which is a path with M'-edges $>$ M-edges.

$\Rightarrow$ This path P is an alternating path for G, M.

(Qn: What about edges in $M \cap M'$?)

$\square$

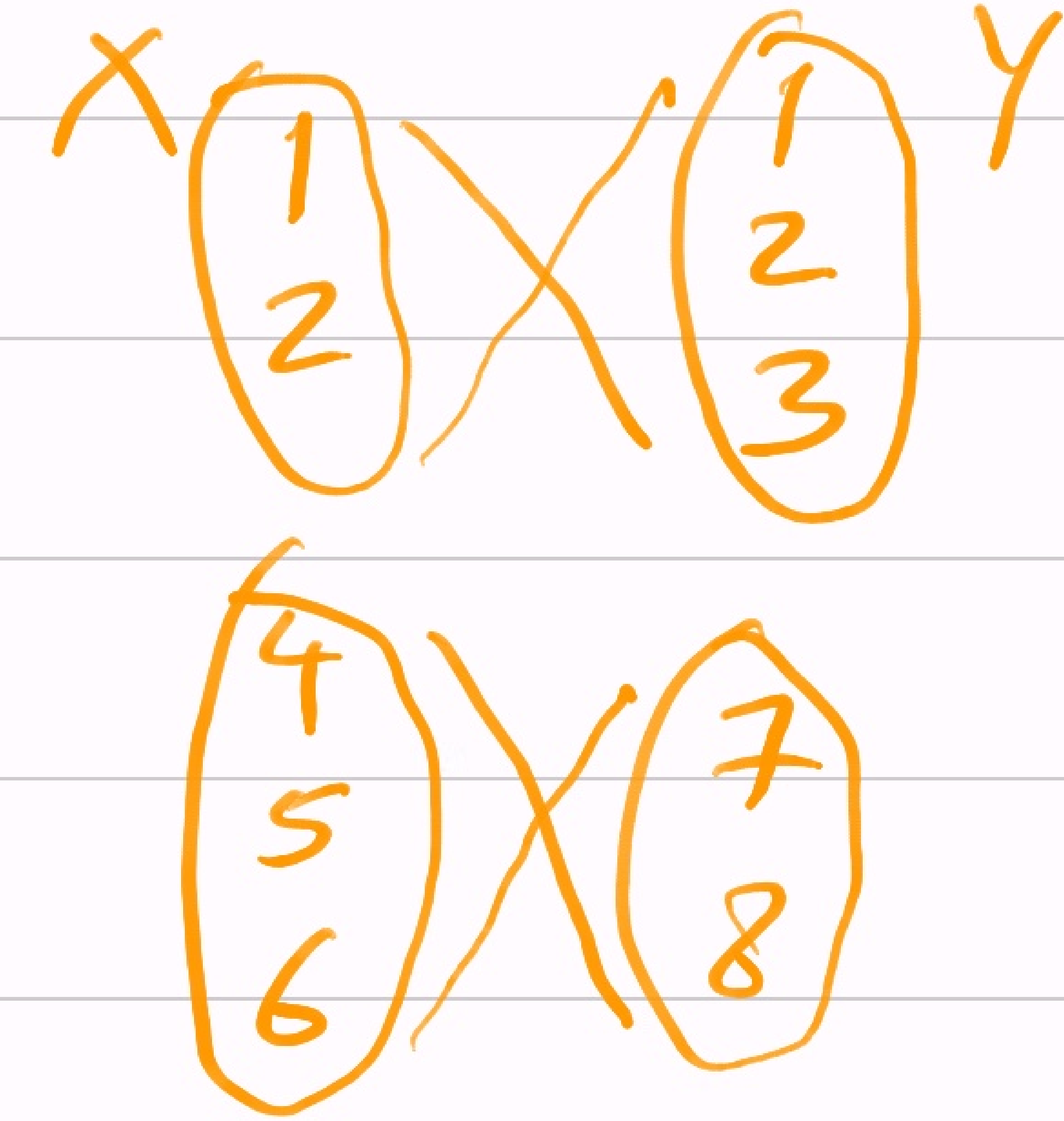
Complete Matching

- Defn: • Let M be a matching in $G=(X\sqcup Y, E)$ with $|X| \leq |Y|$. M is a complete matching if it covers X.
  • If $|X|=|Y|$, then this M is a perfect matching.

- Eg. A bipartite graph with deg > 3 with no complete matching: $G = K_{d,d+1} \sqcup K_{d+1,d}$ $\forall d \geq 0$.

- **Defn:** For $S \subseteq X$, define its <u>neighborhood</u> $N(S) := \{v \in Y \mid \exists u \in S, (u,v) \in E\}$.

G:



▷ Complete matching in $G \implies \forall S \subseteq X, |N(S)| \geq |S|$.

▷ [Hall's condition]

▷ $\{4,5,6\}$ has only 2 neighbors.

<u>Thm</u> [Hall's marriage thm, 1935]: Complete matching in G <u>iff</u> $\forall S \subseteq X, |N(S)| \geq |S|$.

<u>Pf</u>: ($\implies$): $\exists S \subseteq X, |N(S)| < |S| \implies$ <u>No</u> compl. match.

($\impliedby$): $\forall S, |N(S)| \geq |S|$. Let M be a maximum matching that's not complete.

Idea: Show that $\exists$ aug. path for G, M.

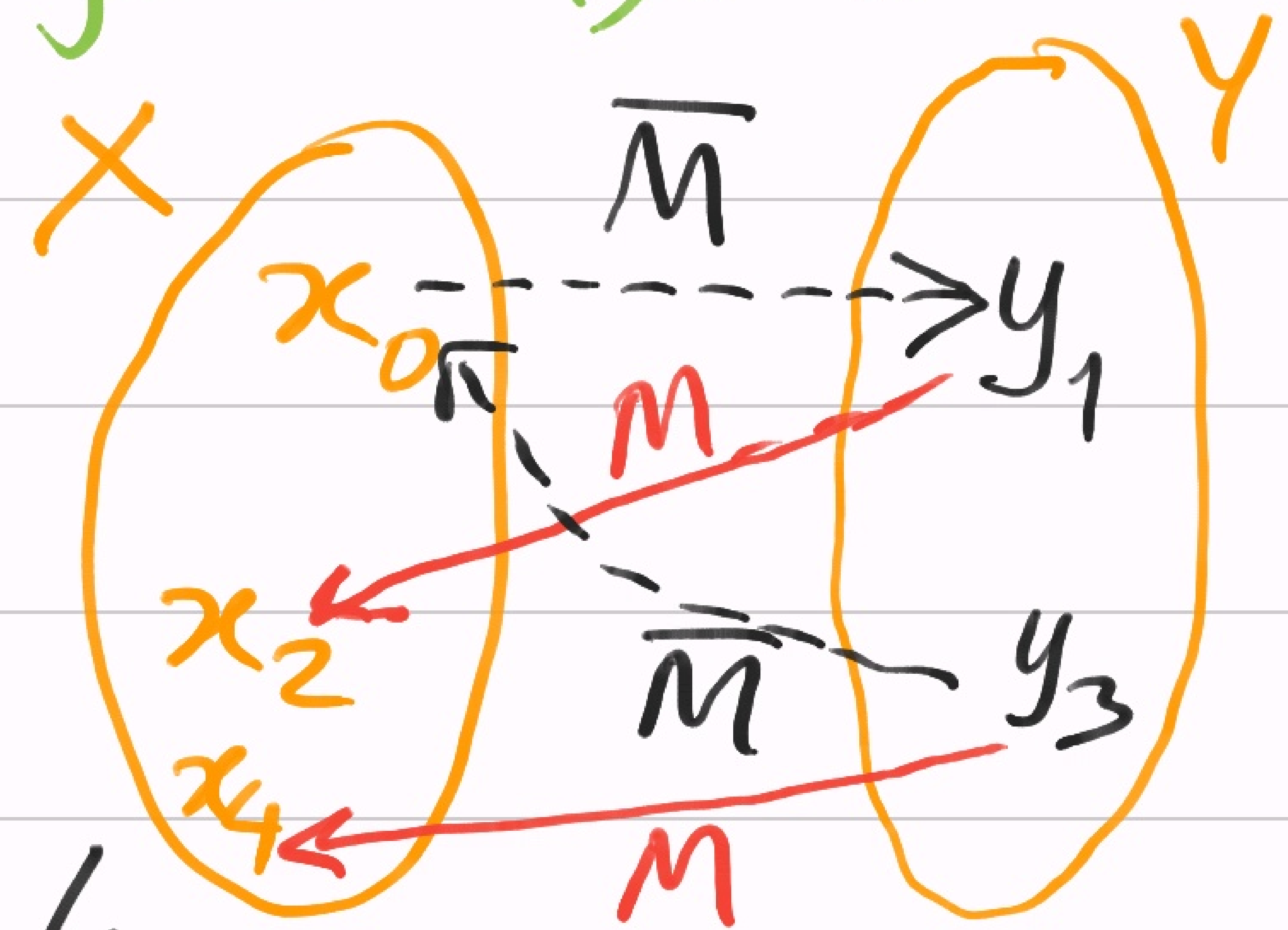
- Let $x_0 \in X$ be unmatched
  (i.e. M-edges not incident on x_0).

$\Rightarrow$ (Hall's condition) $\exists\, y_1 \in N(x_0)$.

Qn: Is y_1 unmatched?

- If it is, then we grow $M \Rightarrow \lightning$.

$\Rightarrow$ (Hall's condition) $\exists\, y_3 \in N(x_0, x_2)$.



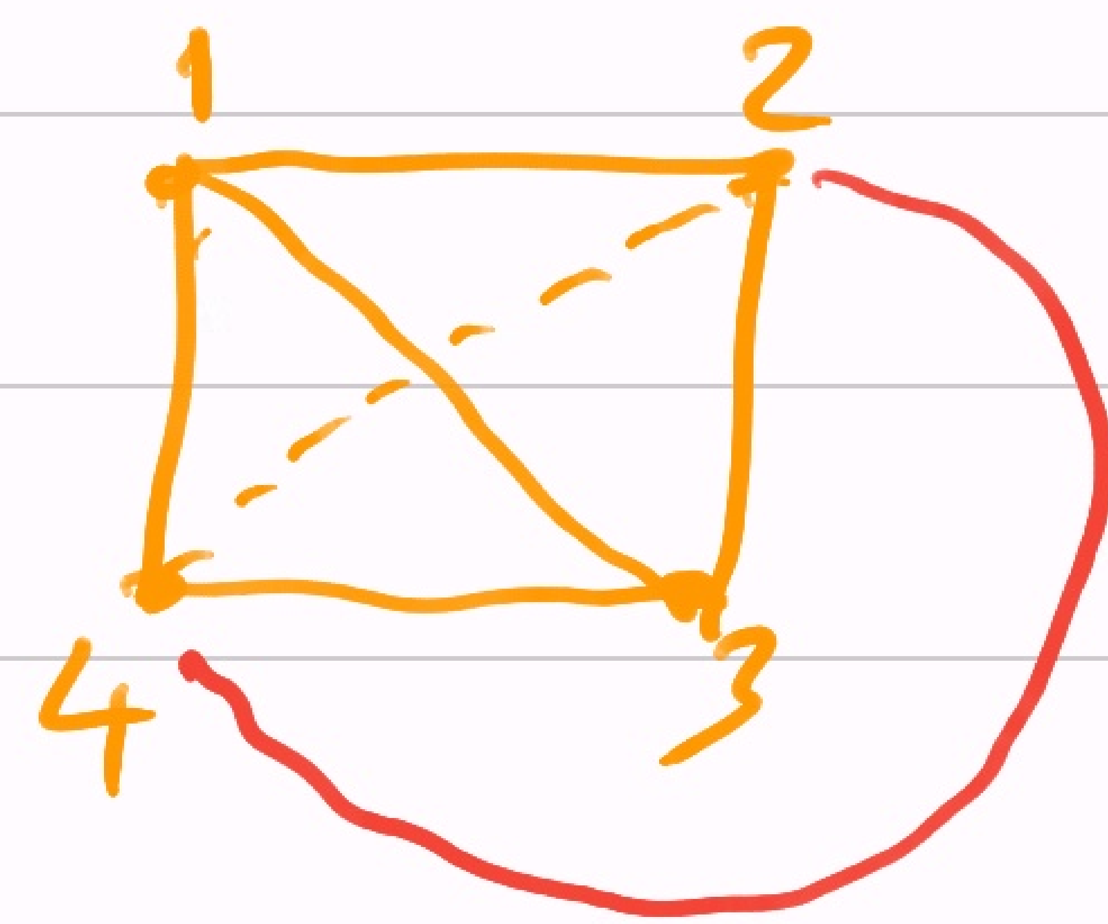
- Case 1 [$(x_2, y_3) \in E$]: $\Rightarrow$ x_0, y_1, x_2, y_3 is aug. path. $\Rightarrow \lightning$
- Case 2 [$(x_0, y_3) \in E$]: $\Rightarrow$ x_0, y_3 is aug. path. $\Rightarrow \lightning$
- Case 3 [$(y_3, x_4) \in M$]: $\Rightarrow$ proceed with the path ... $\Rightarrow \lightning$

$\Rightarrow$ M is complete. $\square$

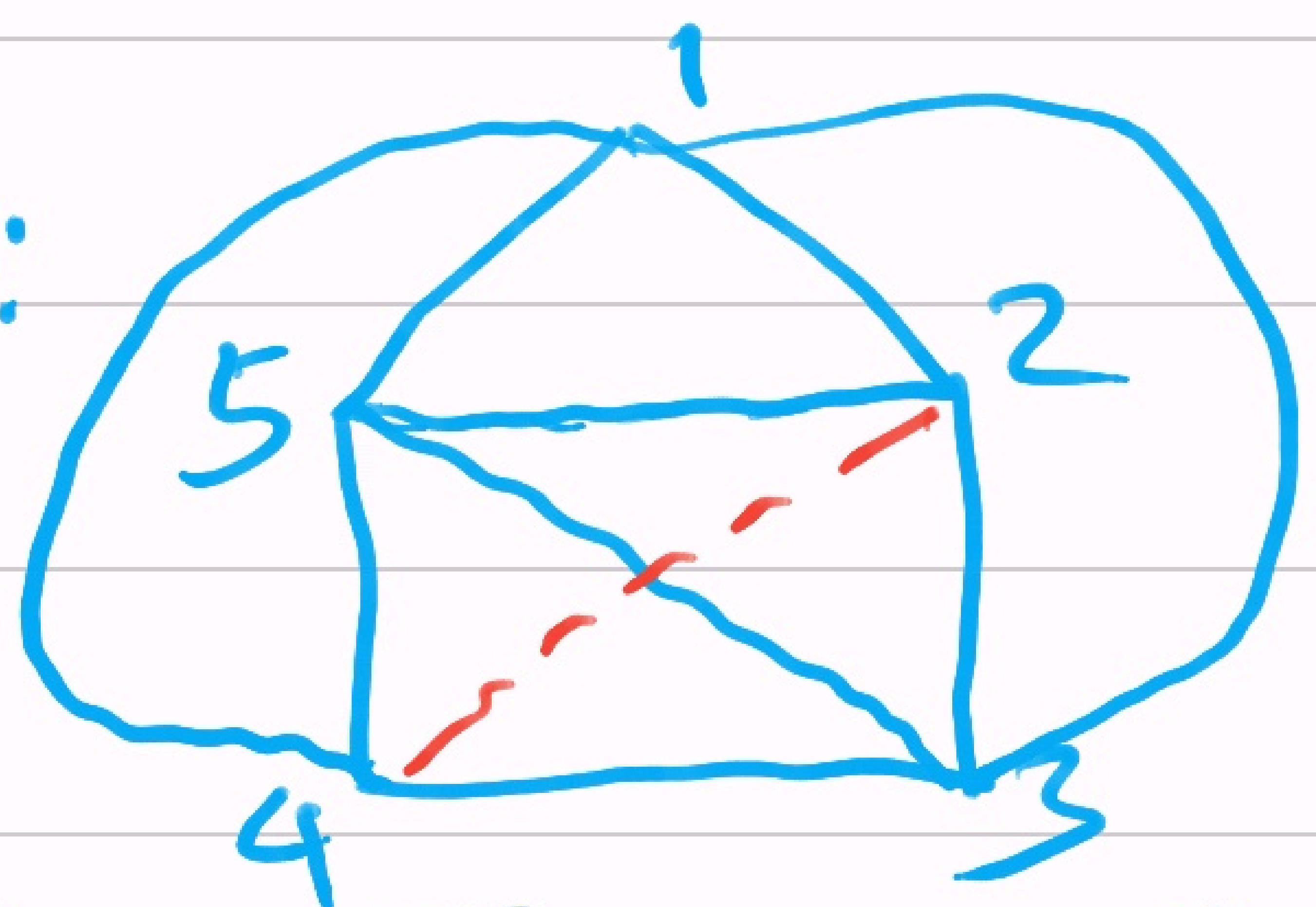
Planar Graphs

- **Def^n**: G is planar if it can be drawn on a paper without two edges intersecting except at vertices.

e.g. K_4:



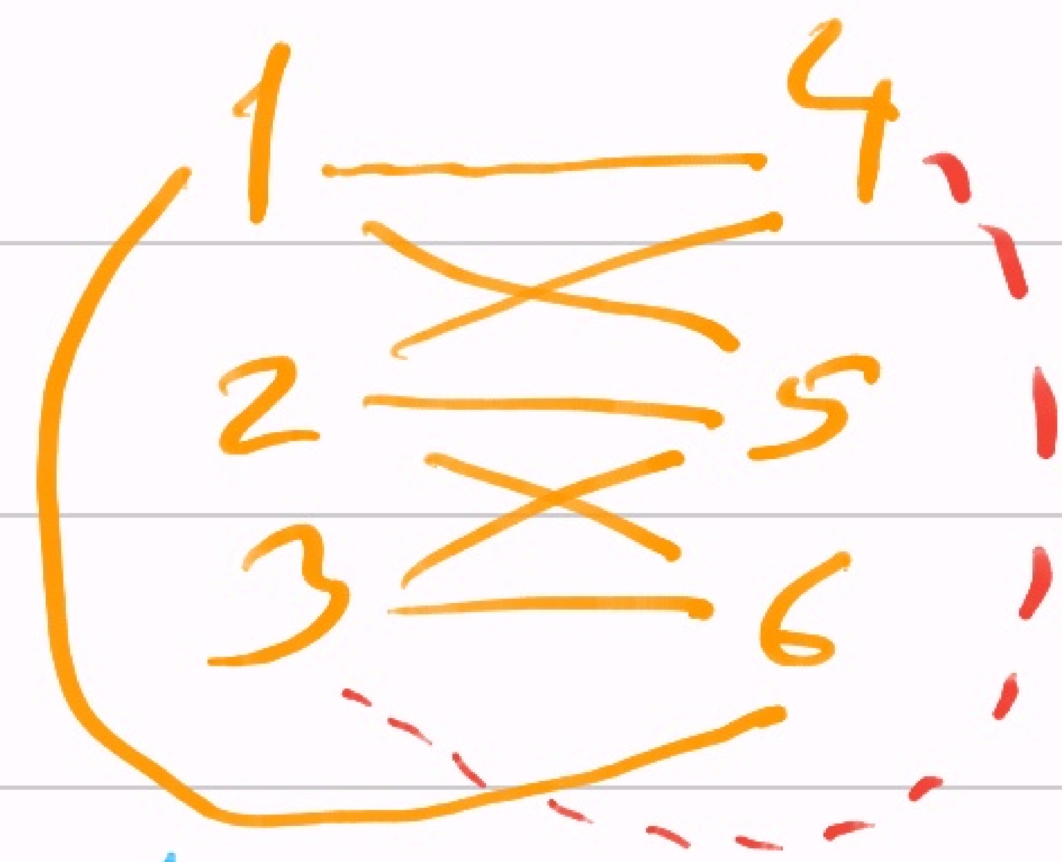
- K_5:



Exercise: K_5 is non-planar.

Major Thm: G planar $\Rightarrow \chi(G) \le 4$.

▷ $K_{3,3}$ is non-planar.

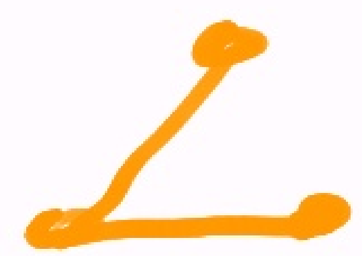


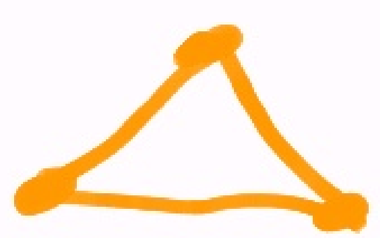
Meta-Thm: G non-planar $\Longleftrightarrow$ either K_5 or $K_{3,3}$ is "embedded".

Theorem (Euler's formula, 1752): G is a connected planar graph $\Rightarrow v - e + f = 2$.

#vertices #edges #faces/regions

eg.  : $1-0+1=2$

 : $2-1+1=2$;  : $3-2+1=2$

 : $3-3+2=2$ Pf: Induction on e. □