Introduction to Combinatorial Game Theory

Abhishek Kumar
Manish Kumar Bera

IITK

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Outline

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2 Few Combinatorial Games
   - Basic terminologies and Strategies
   - Tic-Tac-Toe
   - Hex
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Introduction

Definition

Combinatorial games are two-player games with no hidden information and no chance elements. We will denote two players by $L$ and $R$. 

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Definition

For any game position \( G \) we denote left options of game by \( \mathcal{O}^l \) and right options of game by \( \mathcal{O}^r \).

Thus any game position can be written as

\[
G = \{ \mathcal{O}^l | \mathcal{O}^r \}
\]
Tic-Tac-Toe

Tic-Tac-Toe is a game for two players, X and O, who take turns marking the spaces in a 3x3 grid. The player who succeeds in placing three of their marks in a horizontal, vertical, or diagonal row wins the game.

Strategy in this game is to block the opponent's move try to create a double attack position. For the first player the best opening move would be center position as it gives max. opportunities.

It is very easy to show that if both players play optimally the game always ends in a draw.

Result can be predicted after two initial moves only(one each player), given both play optimally thereafter.
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Tic-Tac-toe is a zero sum game.

There is also a misere version in which one forces the opponent to place three cuts. And there are many variations of the game.
Hex

Hex is a strategy board game played on a hexagonal \( nxn \) grid. One player tries to make a path from top to bottom and other from left to right.

**Figure:** hex board.
Theorem

*Hex can never end in draw.*
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Definition

A (class of) game(s) is determined if for all instances of the game there is a winning strategy for one of the players (not necessarily the same player for each instance).

Proof.

If the second player has a winning strategy, the first player could “steal” it by making an irrelevant move, and then follow the second player’s strategy. If the strategy ever called for moving on the square already chosen, the first player can then make another arbitrary move. This ensures a first player win. Clearly such a strategy cannot exist.
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There are many important points one should keep in mind while playing HEX:

- Prefer two-bridge instead of one.
- Don't block too close to opponent's chain.
- Your chain is as strong as its weakest link.
- Control the center of the board.
- Since first player has advantage, few versions allow second player to swap with first player after first move.
  So choosing first move becomes tricky!
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NIM

Definition

Nim is a strategic game in which there are heaps of coins. Players can take away any number of coins from a particular heap. Player with no legal move loses.

There is also a misère version in which the player who takes the last coin loses.

Both versions are "determined."
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Nim Sum : \( a \oplus b = \) first write \( a \) and \( b \) in binary then add without carrying.

If nim sum of no. of coins in all the heaps is zero then G is called zero position.
Theorem

**Bouton’s theorem:** If \( G \) is a zero position, then every move from \( G \) leads to a nonzero position. If \( G \) is not a zero position, then there exists a move from \( G \) to a zero position.
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Courtesy this theorem we have well defined outcome for every nim position. Also this theorem provides a winning strategy.
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Courtesy this theorem we have well defined outcome for every nim position. Also this theorem provides a winning strategy.

- We know that finally we’ll have 0 coins left which is a zero nim sum position.
- Hence if we start from a zero nim sum position second player will loose and vice-versa.
Fundamental theorem of combinatorial Games

**Theorem**

Let $G$ be a short combinatorial game, and assume normal play. Either Left can force a win playing first on $G$ or else Right can force a win playing second, but not both.

The theorem has an obvious dual, in which “Left” and “Right” are interchanged.

No explicit base case.

The Fundamental Theorem shows that every short game belongs to one of the four normal-play outcome classes $N$, $P$, $L$, $R$. We denote by $o(G)$ the outcome class of $G$. 
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- We denote by $o(G)$ the outcome class of $G$. 
The outcome class of a game, $G$, can be determined from the outcome classes of its options as shown in the following table:

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<th>$G$ $R \in R \cup P$</th>
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**Definition**

A game is impartial if both players have the same options from any position. Else it is called Partisan game.

**Theorem**

If $G$ is an impartial game then $G$ is in either $R$ or $P$. 
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**Definition**

A game is impartial if both players have the same options from any position. Else it is called Partisan game.

**Theorem**

*If $G$ is an impartial game then $G$ is in either $N$ or $P$.***
Theorem

Suppose the positions of a finite impartial game can be partitioned into mutually exclusive sets $A$ and $B$ with the properties:

- every option of a position in $A$ is in $B$
- every position in $B$ has at least one option in $A$.

Then $A$ is the set of $\mathcal{P}$ positions and $B$ is the set of $\mathcal{N}$ positions.
Definition

\[ G + H := \{ G + h^L, H + g^L | G + h^R, H + g^R \} \]

The comma is intended to mean set union.
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**Definition**

\( G = H \) if \( (\forall X) G + X \) has the same outcome class as \( H + X \)

In essence, \( G \) acts as \( H \) in any sum of games.

**Theorem**

1. \( G + 0 = G \)
2. \( G + H = H + G \)
3. \( (G + H) + J = G + (H + J) \)
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Definition

\( -G ::= \{ -\phi_R | -\phi_L \} \)

The definition of negative corresponds exactly to reversing the roles of the two players.
Definition

\[ G - H ::= G + (-H) \]
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### Lemma

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Lemma
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Lemma
= is an equivalence relation.
Theorem

$G = 0$ iff $G$ is a $P$-position. (i.e., $G$ is win for second player)
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### Theorem

Fix games \( G, H, J \)

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\[ G \geq H \text{ if } (\forall X) \text{ Left wins } G + X \text{ whenever Left wins } H + X \]

\[ G \leq H \text{ if } (\forall X) \text{ Right wins } G + X \text{ whenever Right wins } H + X \]
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Lemma

1. \( G \geq H \text{ iff } H \leq G \)
2. \( G \geq H \text{ and } G \leq H \text{ iff } G = H \)
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### Lemma

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2. \( G \geq H \text{ and } G \leq H \text{ iff } G = H \)

### Theorem

*Let \( G \) be any game and let \( Z \in P \) be any game that is a second player win. Then outcome classes of \( G \) and \( G + Z \) are the same.*
Theorem

The following are equivalent:

1. $G \geq 0$. 

These results give us an insight on how to actually compare games $G$ and $H$.
Theorem

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The following are equivalent:

1. $G \geq 0$.
2. Left wins moving second in $G$.
3. For all games $X$, if left wins moving second/first in $X$, then left wins moving second/first on $G + X$.

These results give us an insight on how to actually compare games $G$ and $H$: $G > H$ when $L$ wins $G - H$, $G = H$ when $P$ wins $G - H$. 

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Theorem

The following are equivalent:

1. $G \geq 0$.
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3. $\forall$ games $X$ if left wins moving second/first in $X$ then left wins moving second/first on $G + X$.

Theorem

$G \geq H$ iff $G + J \geq H + J$
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The following are equivalent:

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Theorem

$G \geq H$ iff left wins moving second on $G - H$

These results give us an insight on how to actually compare games $G$ and $H$

- $G > H$ when $L$ wins $G - H$
- $G = H$ when $P$ wins $G - H$
• $G < H$ when R wins $G - H$
• $G || H$ when N wins $G - H$

$||$ means that both games are incomparable.
- $G < H$ when $R$ wins $G - H$
- $G \parallel H$ when $N$ wins $G - H$

$\parallel$ means that both games are incomparable.

**Theorem**

The relation $\geq$ is a partial order on games.

- **Transitive**: $G \geq H$ and $H \geq J$ then $G \geq J$
- **Reflexive**: $G \geq G$
- **AntiSymmetry**: $G \geq H$ and $H \geq G$ then $G = H$
- \( G < H \) when R wins \( G - H \)
- \( G \parallel H \) when N wins \( G - H \)

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**Theorem**

The group containing all games form a partially ordered abelian group under \( + \)
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Every game $G$ has a unique "smallest" game which is equal to it. This game is called $G$’s *canonical form*.
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**Theorem**

If

$$G = \{A, B, C, \ldots | H, I, J, \ldots\}$$

and $B \geq A$ then $G = G'$ where

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Here option $A$ is said to be dominated by option $B$ for Left.
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**Definition**

A Left option $A$ of $G$ can be considered to be reversible if $A$ has a right option $A^R$ such that $A^R \leq G$. 
Theorem

Fix a game

\[ G = \{ A, B, C, ..., | H, I, J, ... \} \]

and suppose for some Right option of \( A \), call it \( A^R \), \( G \geq A^R \). If we denote Left options of \( A^R \) by \( \{ W, X, Y, ... \} \):

\[ A^R = \{ W, X, Y, ... | ... \} \]

and define the new game

\[ G' = \{ W, X, Y, ..., B, C, ... | H, I, J, ... \} \]

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Finally we have our result.

Theorem

If \( G \) and \( H \) are in canonical form and \( G = H \), then \( G \equiv H \) (Isomorphic Games).