Proving Lower Bounds via Pseudo-Random Generators

Manindra Agrawal

IIT Kanpur

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Overview

1. Lower Bounds History
2. Pseudo-Random Generators
3. Applications of Time-Bounded Pseudo-Random Generators
   - Derandomizing Randomized Algorithms
   - Formalizing Cryptographic Security
   - Lower Bounds
4. Lower Bounds on Boolean Circuits
5. Lower Bounds on Arithmetic Circuits

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Approaches to Lower Bounds

- Proving lower bounds on the complexity of problems is the central aim of complexity theory.
- Most important amongst these is to prove $P \neq NP$.
- So far, we have not been very successful.
- Two approaches have been used over last thirty years but both have hit roadblocks.

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First Approach: Diagonalization

Basic Idea

To prove that the set A does not belong to complexity class $\mathcal{C}$.

- Consider the (infinite) sequence of Turing machines accepting precisely the class of sets in $\mathcal{C}$.
- Let this sequence be $M_1, M_2, \ldots$.
- Show that for every i, there is a string x_i that belongs to set A iff M_i rejects x_i.

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- Earliest approach, popular in 1970s.
- Useful for seperating complexity classes that are very "far apart," e.g., P and EXP.
- Did not work for closer classes, e.g., P and NP.
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Example: Seperating P From EXP

- Let M_1, M_2, $\ldots$ be an enumeration of deterministic TMs with M_i running for at most $n^{|i|}$ steps on an input of size n.
- Define a set A as:

$$A = \{i \mid M_i \text{ rejects } i\}.$$

- Set A is in EXP.
- If TM M_j from the above sequence accepts A then M_j accepts j iff M_j rejects j.

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- A family of circuits, one for each input length, corresponds to a set in the class.
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To prove that the set A does not belong to complexity class $\mathcal{C}$.

- Consider the circuit characterization of $\mathcal{C}$.
- This is given by a family of circuits, one circuit for every input length, for each set in $\mathcal{C}$.
- Prove that any circuit on input length n from the families can be transformed to a "simple" circuit that "approximates" the original circuit well.
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- Proposed in 1980s.
- Biggest successes were lower bounds on monotone and constant depth circuit classes.
- Razborov (1985) seperated the class of sets characterized by polynomial sized monotone circuits from the class of sets in NP accepted by monotone circuits.
- Furst-Saxe-Sipser (1984), Håstad (1986) showed that the set PARITY does not belong to the class of sets characterized by constant depth, polynomial sized circuits.

PARITY is the set of all strings that have an odd number of 1's.

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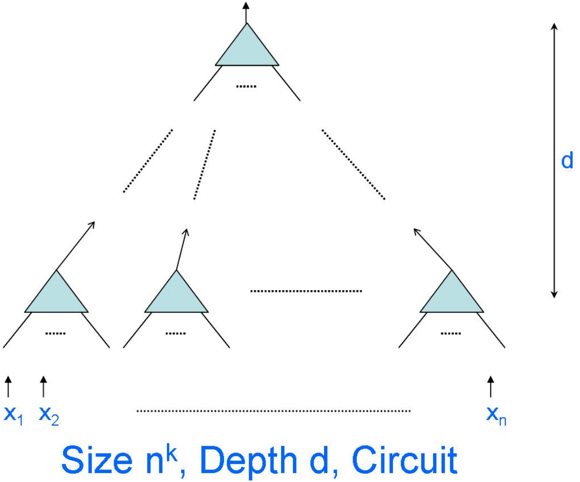
Example: Lower Bounds on PARITY



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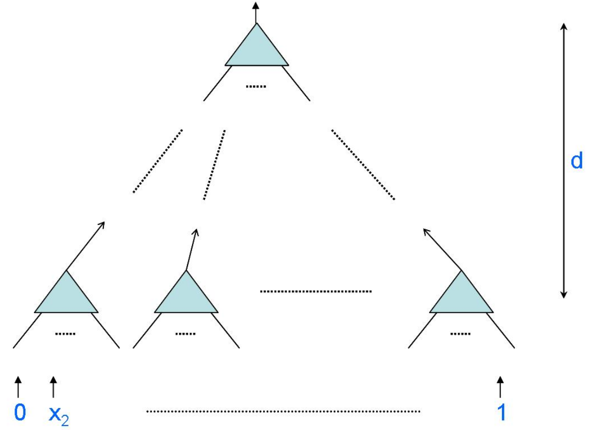

Random Assignment to n-n$^{\delta}$ Input Bits

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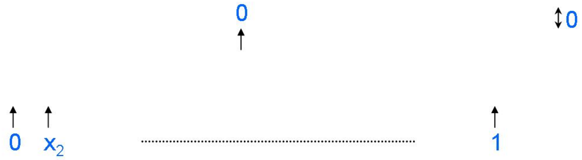

Reduces to Fixed Circuit with Prob > 0

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- Appeared very promising in the beginning.
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- They classified the combinatorial arguments used as natural proofs.
- And showed, under very reasonable assumptions, that no natural proof can prove lower bounds on circuit classes significantly larger than constant depth, polynomial sized.

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A New Approach: Pseudo-Random Generators

- Pseudo-random generators were defined in 1980s for two reasons:
  - To formalize the notion of cryptographic security.
  - To derandomize probabilistic algorithms.
- In 1990s, they were shown to be equivalent to certain types of lower bounds.
- Recently, there are indications that they might be useful in proving lower bounds.

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Definition

Let $\mathcal{C}(n, d)$ be the class of depth d, size n boolean circuits on n inputs.

Let $f : \{0,1\}^* \mapsto \{0,1\}^*$ be a function such that $|f(y)| = n$ for all strings y of length $\ell(n) < n$.

Definition

Function f is a $(\ell(n), n)$-pseudo-random generator against $\mathcal{C}(n, d)$ if for every circuit $C \in \mathcal{C}(n, d)$,

$$\frac{1}{2^n} \mid \{x \mid C(x) = 1\} \mid - \frac{1}{2^{\ell(n)}} \mid \{y \mid C(f(y)) = 1\} \mid \leq \frac{1}{n}.$$

String y is called the seed, and the difference $n - \ell(n)$ is called the stretch of the generator.

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Existance of Pseudo-Random Generators

Let C be any circuit in $\mathcal{C}(n, n)$. Define F as: On input y, $|y| = 5 \log n$, output a random string of length n.

- For any y, define random variable Z_y as: $Z_y = C(f(y))$.
- Then,

$$\sum_y Z_y = \mid \{y \mid C(f(y)) = 1\} \mid .$$

- And,

$$\Pr[Z_y = 1] = \frac{1}{2^n} \mid \{x \mid C(x) = 1\} \mid = \mu_C \text{ (say)}.$$

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- By Chernoff's bound:

$$\Pr[|\frac{1}{n^5}\sum_y Z_y - \mu_C| > \delta\mu_C] < e^{-n^5\mu_C\delta^2/4} < e^{-n^5\delta^2/4}.$$

- Choosing $\delta = \frac{1}{n}$, we get:

$$\Pr[|\frac{1}{n^5}\sum_y Z_y - \mu| > \frac{1}{n}] < e^{-n^3/4}.$$

- Since there are less than 2^{n^2} circuits in $\mathcal{C}(n,n)$, probability that F fails to approximate μ_C for some $C \in \mathcal{C}(n,n)$ is at most $\frac{1}{2^{n/4}}$.
- Hence, most of the functions from $\{0,1\}^{5\log n}$ to $\{0,1\}^n$ are pseudo-random against $\mathcal{C}(n,n)$.

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Optimal Pseudo-Random Generators

Function f is an optimal pseudo-random generator against $\mathcal{C}(n, d)$ if it is a $(O(\log n), n)$-pseudo-random generator against $\mathcal{C}(n, d)$.

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Time-Bounded Pseudo-Random Generators

An $(\ell(n), n)$-pseudo-random generator f is $t(m)$-computable if there is a $t(m)$-time bounded DTM that, on input (y, j), $|y| = m = \ell(n)$ and $1 \leq j \leq n$, outputs the jth bit of $f(y)$.

Time-bounded pseudo-random generators are very interesting!

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Derandomizing BPP

Suppose there exists a $2^{O(m)}$-computable optimal pseudo-random generator f against $\mathcal{C}(n, n)$.

- Let $\mathcal{B}$ be a randomized polynomial-time algorithm accepting a set B in BPP.
- View $\mathcal{B}$ as taking two inputs x and r, with x being the "real" input and r being a sequence of random bits.
- Assume that $|r|$ equals the square of time taken by $\mathcal{B}$ on input x.

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- Fix any x. Then $\mathcal{B}(x, r)$ can be thought of as a circuit C of size $n = |r|$ operating on input r.
- Circuit C outputs a 1 on either at least $\frac{2}{3}$-fraction or at most $\frac{1}{3}$-fraction of these inputs depending on whether x is in the set B or not.
- Therefore, C will output a 1 on either at least $(\frac{2}{3} - \frac{1}{n})$-fraction or at most $(\frac{1}{3} + \frac{1}{n})$-fraction of inputs of the form $f(y)$.

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- Since f is optimal, $|y| = O(\log n)$.
- Since f is $2^{O(m)}$-computable and $m = |y| = O(\log n)$, $f(y)$ can be computed in time $n^{O(1)}$.
- Therefore, in time polynomial in n, one can deterministically decide if x is in the set B or not.
- Since $n = |r|$, n is a polynomial in $|x|$.
- This shows that $B \in \mathrm{P}$.
- Thus, $\mathrm{BPP} = \mathrm{P}$.

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- Define function g as: on input y, $|y| = m$, output the first m^4 bits of $f(y)$.
- Function g is efficiently computed since first m^4 bits of f can be computed in time $m^{O(1)}$.

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Lower Bounds via Pseudo-Random Generators

Suppose there exists a $2^{O(m)}$-computable optimal pseudo-random generator f against $\mathcal{C}(n, n)$.

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Equivalence of Lower Bounds and Pseudo-Random Generators

Theorem (Håstad-Impagliazzo-Levin-Luby (1990))

There exist $m^{O(1)}$-computable $(n^{o(1)}, n)$-pseudo-random generators against $\mathcal{C}(n, n)$ iff there exist one-way functions.

One-way functions are functions computable in polynomial-time whose inverse is hard-to-compute.

Equivalence of Lower Bounds and Pseudo-Random Generators

Theorem (Impagliazzo-Wigderson,1997)

There exist $2^{O(m)}$-computable optimal pseudo-random generators against $\mathcal{C}(n,n)$ iff there exist sets in E that cannot be computed by subexponential-sized circuit family.

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Why Should Pseudo-Random Generators be Any Easier to Construct?

Pseudo-random generators avoid natural proof block.

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Some techniques in circuit model are known to be non-relativizable, e.g., Håstad's Switching Lemma.

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The problem is of designing an algorithm.

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There are a number of derandomization primitives available, e.g., extractors, expanders, pairwise independence.

- Expander graphs were recently used by Reingold (2005) to derandomize searching in undirected graphs proving SL = L.

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First Step: Against Constant Depth Circuits

- Håstad (1986) proved that PARITY cannot be accepted by depth d circuits of size $2^{n^{1/14d}}$.
- By Nisan-Wigderson (1987), this yields a $m^{O(1)}$-computable, $(\log^{O(d)} n, n)$-pseudo-random generator against $\mathcal{C}(n, d)$.
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Step 1.

For each $d > 0$, construct a $2^{O(m)}$-computable optimal pseudo-random generator against $\mathcal{C}(n, d)$.

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There exists a $2^{O(m)}$-computable optimal pseudo-random generator against $\mathcal{C}(n, d)$

$$\Downarrow$$

There is a set B in E that cannot be accepted by any subexponential sized depth d circuit family

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B cannot be accepted by any $n^{d-\epsilon}$ size, $(d - \epsilon)\log n$ depth circuit family with bounded fanin AND gates for any $\epsilon > 0$

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Step 3.

Construct a $m^{O(1)}$-computable optimal pseudo-random generator against $\mathcal{C}(n, \log n)$.

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- Although the increase in depth is small, it improves the lower bound enormously because of inherent exponentiation.
- The generator implies that NP cannot be accepted by any family of sublinear depth and subexponential sized circuits.
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Fourth Step: Further Enlarge the Class of Circuits

Step 4.

Construct a $m^{O(1)}$-computable optimal pseudo-random generator against $\mathcal{C}(n, \log^{O(1)} n)$.

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- Again, because of exponentiation, this implies that NP cannot be accepted by any family of polynomial depth and subexponential sized circuits.
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- We known $m^{O(1)}$-computable optimal pseudo-random generator against $\mathcal{C}(n,2)$, the class of depth two circuits.
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Arithmetic Circuits

- Arithmetic circuits over field F are circuits with addition, subtraction, and multiplication gates.
- These compute a polynomial over the field F.
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- For example, computing determinant, finding roots of a polynomial, finding short vectors in a lattice etc.

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Power of Arithmetic Circuits

- Polynomial sized arithmetic circuits can solve all the above problems.
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- The converse is unlikely as shown by Valiant et. al. (1983):
  - A polynomial sized arithmetic circuit of polynomial degree can be transformed to polynomial sized arithmetic circuit of logarithmic depth and fanin two multiplication gates.

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Identity Testing and Lower Bounds

- Identity Testing problem is that given a polynomial computed by an arithmetic circuit, test if the polynomial is identically zero.
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Pseudo-Random Generators Against Arithmetic Circuits

Let $\mathcal{A}(n, F)$ be a subclass of size n arithmetic circuits over field F.

Let $f : \mathbb{N} \mapsto (F[y])^*$ be a function such that $f(n) = (f_1(y), \ldots, f_n(y), g(y))$ for all n.

Pseudo-Random Generators Against Arithmetic Circuits

Function f is an efficiently computable optimal pseudo-random generator against $\mathcal{A}(n, F)$ if

- Each $f_i(y)$ and $g(y)$ is of degree $n^{O(1)}$.
- Each $f_i(y)$ and $g(y)$ is computable in time $n^{O(1)}$.
- For any circuit $C \in \mathcal{A}(n, F)$ with $m \leq n$ inputs:

$$C(x_1, x_2, \ldots, x_m) = 0 \text{ iff } C(f_1(y), f_2(y), \ldots, f_m(y)) = 0 \ (mod\ g(y)).$$

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- If there exist efficiently computable optimal pseudo-random generators against the entire class of size n circuits then:
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- Define polynomial q as:

$$q(x_1, x_2, \ldots, x_{2m}) = \sum_{S \subseteq [1,m]} c_S \prod_{i \in S} x_i.$$

- Here $c_S \in F$ satisfying:

$$\sum_{S \subseteq [1,m]} c_S \prod_{i \in S} f_i(y) = 0.$$

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Why Should Pseudo-Random generators be Easier to Construct?

- A-Kayal-Saxena (2002) constructed an efficiently computable optimal pseudo-random generator against a very special class of circuits.
- This contained circuits computing the polynomial $(1+x)^m - x^m - 1$ over ring Z_m.
- The pseudo-random generator was:

$$f(n) = (x, x, \ldots, x, g(x)), g(x) = x^{16n^5} \prod_{r=1}^{16n^5} \prod_{a=1}^{4n^4} ((x-a)^r - 1).$$

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A Possible Way of Proving Hardness of Permanent

- The complexity of computing permanent of a matrix characterizes the class #P.
- #P is the arithmetic analog of the class NP.
- We give a stepwise approach to prove hardness of permanent.
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First Step: Against Constant Depth Circuits

Step 1.

For each $d > 0$, construct an efficiently computable optimal pseudo-random generator against the class of size n, depth d arithmetic circuits.

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There exists an efficiently computable optimal pseudo-random generator against the class of size n, depth d arithmetic circuits

$$\Downarrow$$

There is a multilinear polynomial q computable in PSPACE that cannot be computed by subexponential sized, depth d circuits

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- The union over all d's spans all polynomial sized circuits!
- This motivates the second step.

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Construct an efficiently computable optimal pseudo-random generator against the class of size n, depth $\omega(1)$ arithmetic circuits.

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- Suppose each coefficient of the hard-to-compute multilinear polynomial given by a generator can be computed by a #P-function.
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Such a generator implies that Permanent requires superpolynomial sized arithmetic circuits.

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- We know efficiently computable optimal pseudo-random generators against size n, depth two arithmetic circuits.
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THANK YOU!