Quasi-linear Truth-table Reductions to P-selective Sets *

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Abstract

We show that if SAT is quasi-linear truth-table reducible to a p-selective set then NP = P. As a consequence it follows that for a class $\mathcal{K} \in \{\mathrm{PP}, \mathrm{C_=P}\}$, if every set in $\mathcal{K}$ is quasi-linear truth-table reducible to a p-selective set then $\mathcal{K} = \mathrm{P}$.

1 Introduction

The study of reductions[1] of complexity classes to sets of low information content is central in structural complexity theory. Among different notions of low information content, *sparseness* has received much attention over the years. Another such notion is *p-selectivity*, introduced by Selman [Sel79]. It is inspired by the semi-recursive sets of recursive function theory [Jok68]. P-selectivity is a complexity-theoretic generalization of computable real numbers [Sel79, Sel81, Ko83]. Actually p-selectivity turned out to be related to sparseness. The class of sets Turing reducible to p-selective sets is precisely the class P/poly, of sets accepted by nonuniform polynomial-size circuits [Sel79]. The class of sets Turing-reducible to sparse sets is also P/poly [KL80, KL82].

A research trend is to seek for strong collapses of complexity classes, as a consequence of the assumption that they are reducible to low information content sets. For example, in the case of sparse sets, an important result is the extension of Mahaney's theorem [Mah82] to bounded truth-table reductions: if NP is bounded truth-table reducible to a sparse set then NP = P [OW91]. Similar research concerning reductions to p-selective sets has been reported.

It was shown in [Sel82] that if NP (indeed any disjunctive self-reducible set) is positive truth-table reducible to a p-selective set then NP = P (or that disjunctive self-reducible set is in P). This result has been recently extended in [Bu93]. They show that if a Turing self-reducible set is (in the sense of Ko [Ko83]) positive-Turing reducible to a p-selective set then it is in P.

*A preliminary version was presented at the Structure in Complexity Theory Conference, 1994 [AA94].
[1] In this paper we consider only polynomial-time computable reductions.

In [Bei88, Tod91], it is shown that if NP (respectively UP) is truth-table reducible to a p-selective set then NP = RP (respectively UP = P). It is also shown that if PSPACE is truth-table reducible to a p-selective set then PSPACE = P.

In [TTW93], it is shown that if NP is bounded truth-table reducible to a p-selective set then SAT $\in DTIME(2^{n^{1/\sqrt{(\log n)}}})$. It is based on a clever recursive use of the fact that if NP is truth-table reducible to a p-selective set implies UP = P [Tod91]. In [HHO$^+$93] it is show that if NP is 1-truth-table reducible to p-selective sets then NP = P.

In this paper we show that if SAT is quasi-linear truth-table reducible to a p-selective set then NP = P. This follows as a consequence of a more general result about disjunctive self-reducible sets. Using standard arguments it also follows that for any class $\mathcal{K} \in \{\mathrm{NP}, \mathrm{PP}, \mathrm{C_{=}P}\}$ that if every set in $\mathcal{K}$ is quasi-linear truth-table reducible to a p-selective set then $\mathcal{K} = \mathrm{P}$.

In the proof of our main result we make use of the linear order on the queries made to a p-selective set. The proof also hinges on the disjunctive self-reduciblity property of SAT and the fact that NP-complete sets have OR functions (defined in Section 3).

Our collapse consequence results for p-selective sets have been obtained independently and at the same time by Beigel et. al. [BKS94] and Ogihara [O94]. Whereas we directly consider the problem of reductions to p-selective sets, Beigel et. al. [BKS94] and Ogihara [O94] consider reductions to membership comparable sets and prove more general results.

Both in our proof and the proofs of [BKS94, O94], OR functions for a set play an important role. The use of OR functions is more implicit and clever in [BKS94, O94]. As a consequence, their proofs are more elegant than ours. Nevertheless, the proof described in the present paper is of interest since it has a different flavor than theirs.

2 Definitions

Strings are over $\Sigma = \{0, 1\}$. For a string $x \in \Sigma^*$, $|x|$ denotes its length. For a finite subset X of Σ^*, $||X||$ denotes the cardinality of X.

Definition 2.1 A set $A \subseteq \Sigma^*$ is *p-selective* if there is a polynomial time function f, $f : \Sigma^* \times \Sigma^* \mapsto \Sigma^*$, such that for every x and y,

1. $f(x, y) \in \{x, y\}$.
2. If $x \in A$ or $y \in A$ then $f(x, y) \in A$.

The function f is called a *p-selector* for A.

For the rest of the section we fix A to be some p-selective set different from $\emptyset$ and Σ^*, and we discuss some basic properties of p-selective sets.

A p-selector f for the set A imposes the following linear ordering on a quotient of Σ^* [Ko83]: Let $x \leq_f y$ if $f(x, y) = x$, define $\preceq_f$ to be the transitive closure of $\leq_f$, and $x \cong_f y$ iff $x \preceq_f y$ and $y \preceq_f x$. Now, $\preceq_f$ induces a linear ordering on $\Sigma^*/\cong_f$ such that A is the union of an initial segment of this ordering. Define the partial ordering $\prec_f$ as: $x \prec_f y \Leftrightarrow x \preceq_f y \bigwedge x \not\cong_f y$. For technical reasons, it is convenient to introduce a minimum and a maximum element, denoted as $\perp$ and $\top$ respectively, such that for every $x \in \Sigma^*$, $\perp \prec_f x \prec_f \top$. The following proposition guarantees that $\perp$ and $\top$ can be introduced for every p-selective set.

Proposition 2.2 *For every p-selective set A there exists a p-selector f for A and strings $\perp, \top \in \Sigma^*$ such that for every $x \in \Sigma^*$, $\perp \prec_f x \prec_f \top$.*

Proof. Let A be p-selective with function g as the p-selector. Let $\perp = x_0$, $\top = y_0$, for some $x_0 \in A$, $y_0 \notin A$ (the case when A is $\emptyset$ or Σ^* can be easily handled separately). Now, we define the new p-selector f as follows: $f(\perp, y) = f(y, \perp) = \perp$, $f(\top, y) = f(y, \top) = y$, and for all $x, y \in \Sigma^* - \{\perp, \top\}$ $f(x, y) = g(x, y)$. ■

For any finite set $Q \subseteq \Sigma^*$, one can modify the above ordering as follows [Tod91]: For all x, $y \in Q$, $x \preceq_{f,Q} y$ iff there exist $z_1, \ldots, z_n \in Q$ such that $z_1 = x, z_n = y$ and $f(z_i, z_{i+1}) = z_i$ for $1 \leq i \leq n-1$.

Clearly, $x \preceq_{f,Q} y \Rightarrow x \preceq_f y$. Define $x \cong_{f,Q} y$ iff $x \preceq_{f,Q} y$ and $y \preceq_{f,Q} x$. This is an equivalence relation on Q and $\preceq_{f,Q}$ induces a linear ordering on the quotient $Q/\cong_{f,Q}$. Define the partial ordering $\prec_{f,Q}$ as: $x \prec_{f,Q} y \Leftrightarrow x \preceq_{f,Q} \bigwedge x \not\cong_{f,Q} y$.

We omit the subscript f when it is clear from the context. Also, when we consider a finite set Q under the $\preceq_{f,Q}$ ordering we implicitly mean the quotient $Q/\cong_{f,Q}$.

It is easy to see that for any finite set Q, the relations $\preceq_Q$, $\prec_Q$ and $\cong_Q$ can be computed in time polynomial in $\sum_{x \in Q} |x|$. Furthermore, the set $A \cap Q$ is an initial segment of Q with respect to $\preceq_Q$.

Definition 2.3 We say that u is a *cut point of* Q if $u \in Q \cup \{\perp, \top\}$, $u \in A$, and for every element $w \in Q$ such that $u \prec_Q w$, $w \notin A$.

Clearly every finite set Q has a cut point. Observe that Q has a unique cut point upto equivalence under $\cong_Q$.

Proposition 2.4 *Let A be a p-selective set and f be a p-selector for A. If u is the cut point of a finite set Q w.r.t. A and f, then we have:*

$$
\begin{aligned}
Q \cap A &= \{w \in Q \mid w \preceq_Q u\} \\
Q \cap \bar{A} &= \{w \in Q \mid u \prec_Q w\}.
\end{aligned}
$$

Definition 2.5 [LLS75] A set B is *truth-table reducible* to a set A, denoted $B \leq_{tt}^p A$, if there are two polynomial time functions, g and e satisfying the following conditions.

- On input $x \in \Sigma^*$ the *generator* g outputs a set of strings $g(x) = \{q_1, \ldots, q_m\}$.

  Let $\chi_A(g(x))$ denote the m-bit vector such that the i^{th} bit of $\chi_A(g(x))$ is 1 iff $q_i \in A$, $1 \leq i \leq m$.

- The *evaluator* e, given x and $\chi_A(g(x))$ as input, decides the membership of x in B. That is, for any $x \in \Sigma^*$, it holds that $x \in B \Leftrightarrow e(x, \chi_A(g(x))) = 1$.

For any $b(n) \geq 0$, set B is said to be *$b(n)$-truth-table reducible* to A, $B \leq_{b(n)-tt}^p A$, if the generator g outputs at most $b(n)$ strings for each input of length n. If $B \leq_{k-tt}^p A$ for some constant $k \geq 0$, then B is said to be *bounded truth-table reducible* to A, $B \leq_{btt}^p A$. If $B \leq_{O(n^{1-\epsilon})-tt}^p A$ for $1 > \epsilon > 0$ then B is said to be *quasi-linear truth-table reducible* to A.

3 The results

Definition 3.1 [Ko83] An irreflexive partial order $\sqsubset$ on Σ^* is *polynomially related* if there is a polynomial p such that

1. $x \sqsubset y$ implies $|x| \leq p(|y|)$,

2. $x \sqsubset y$ is decidable in time polynomial in $|x| + |y|$, and

3. $x_1 \sqsubset x_2 \sqsubset \cdots \sqsubset x_k$ implies $k \leq p(|x_k|)$.

A set L is *disjunctive self-reducible* if there is a polynomial-time oracle machine M such that $L = L(M, L)$, and on input x, M generates queries $y_1, y_2, \cdots, y_m$ and accepts x iff for some i, $1 \leq i \leq m$, $y_i \in L$, where $y_i \sqsubset x$ for each i.

A set L is said to have OR_ω if there is a function OR_ω mapping *finite* subsets of Σ^* into Σ^* such that: for all finite subsets X of Σ^*, $\mathrm{OR}_\omega(X) \in L$ iff $X \cap L \neq \emptyset$. Furthermore, it is required that $\mathrm{OR}_\omega(X)$ is computable in time polynomial in $\sum_{x \in X} |x|$.

In this section we assume that L is a disjunctive self-reducible set with OR_ω and $L \leq_{tt}^p A$ with generator g and evaluator e where A is a p-selective set. For any string x let the queries in $g(x)$ be ordered under $\preceq_{g(x)}$ (it can be done in time polynomial in x). Let $g(x) = \{q_1, \ldots, q_m\}$, $q_0 = \bot$, and $q_{m+1} = \top$ be the $\preceq$-ordered set. The following lemma is obvious from Proposition 2.4.

Lemma 3.2 *For any x, $\chi_A(g(x))$ is in $\{0^m, 10^{m-1}, 110^{m-2}, \ldots, 1^m\}$ where $||g(x)|| = m$.*

Definition 3.3 A string $q \in \Sigma^*$ is called a *true point of x* if there is an i, $0 \leq i \leq m$, such that $q_i \preceq_{g(x)} q \prec_{g(x)} q_{i+1}$ and $e(x, 1^i 0^{m-i}) = 1$. Similarly, a string q is called a *false point of x* if there is an i, $0 \leq i \leq m$, such that $q_i \preceq_{g(x)} q \prec_{g(x)} q_{i+1}$ and $e(x, 1^i 0^{m-i}) = 0$.

The next proposition is immediate.

Proposition 3.4 *$x \in L$ iff the cut point of $g(x)$ is a true point of x.*

We now prove the main theorem of the paper.

Theorem 3.5 *Let L be a set satisfying the following properties:*

1. *L is disjunctive self-reducible.*

2. *L has OR_ω such that for all finite subsets X of Σ^* $|\mathrm{OR}_\omega(X)| = O((\sum_{x \in X} |x|)^l)$, for a constant $l \geq 1$.*

If L is $O(n^{(1/l)-\epsilon})$ truth-table reducible to a p-selective set, for an ϵ such that $0 \leq \epsilon \leq 1/l$, then $L \in \mathrm{P}$.

Proof. Let L be a set satisfying the above properties such that $L \leq_{O(n^{(1/l)-\epsilon})-tt}^p A$, for a p-selective set A. We will give a polynomial-time decision procedure for L.

First we give an intuitive description of the decision procedure. Let x be the input string to be checked for membership in L. The depth of the self-reduction tree rooted at x is bounded by $p(|x|)$ for some polynomial p. We note that, by definition, $x \in L$ iff one of its immediate children in the tree is in L. Extending this property we propose to give a breadth-first pruning algorithm that works in $p(|x|)$ stages. At the i^{th} stage it maintains a list $F = \{x_1, x_2, \ldots, x_s\}$ of strings at depth i in the self-reduction tree for x, with the properties:

- $x \in L$ iff $F \cap L \neq \emptyset$.

- $||F||$ is bounded by a suitable polynomial in $|x|$.

If there is a string in the list F that is a leaf of the self-reduction tree we can directly test for membership in polynomial time, using the self-reducing machine for L. If the leaf-level string is in L then we accept the input x and stop. If the leaf-level string is not in L then we discard it from the list. After this, the algorithm goes to the $i + 1$th stage by replacing each string in F by the set of its children in the self-reduction tree. The list F is then pruned to a polynomially bounded size using a pruning procedure such that the above properties are preserved. In this way it is ensured that if $x \in L$ then at some stage a leaf-level string $y \in L$ will get included in F and membership of x in L will be correctly detected.

We first describe the overall decision procedure DECIDE. The crux of procedure DECIDE is a pruning step that at any stage i preserves the properties explained above. We give the description of this pruning step in procedure PRUNE.

procedure DECIDE(x);
(* DECIDE(x) decides the membership in L of the input string x *)

$F := \{x\}$;
$ACCEPT := false$;
(* Let $p(|x|)$ bound the lengths of all strings in the self-reduction tree rooted at x,
the number of children of any string in the self-reduction tree rooted at x,
as well as the depth of the tree for a suitable polynomial p. *)
for $d := 1$ **to** $p(|x|)$ **do**
  F:=PRUNE(F);
    for every $y \in F$ **do**
      replace y in F by the set of children of y in the self-reduction tree;
        if y is a leaf node in the self-reduction tree **then**
          if $y \in L$ **then** $ACCEPT := true$
        (* Note that for a leaf node y membership testing in L can be done
        in polynomial time. *)
    end-for
end-for;
if $ACCEPT = true$ **then** **return**(ACCEPT) **else** **return**(REJECT)

procedure PRUNE(X);
(* PRUNE(X) returns a subset Y of X of size bounded by $N^{1/l\epsilon}$ such that
$X \cap L \neq \emptyset$ iff $Y \cap L \neq \emptyset$. Here N is the maximum of the lengths of all strings in X *)

$Q := \bigcup_{x \in X} g(x)$;
$\mathcal{I} := \{p_1, p_2, \cdots, p_r\}$, where $\{p_1, p_2, \cdots, p_r\}$ is an ordered list of representatives
  from the equivalence classes of Q induced by $\equiv_Q$;
  repeat
    repeat
      $m := ||X||$;
      for $p \in \mathcal{I}$ **do** $X_p := \{x \mid x \in X$ and p is a true point of $x\}$;
        if $\exists x \in X\ \forall p \in \mathcal{I}$: $x \in X_p \Rightarrow ||X_p|| > 1$ **then** $X := X - \{x\}$
    until $m = ||X||$;
  (* At this stage, for every $x \in X$ there is a $p \in \mathcal{I}$ such that $X_p = \{x\}$. *)
    Construct an ordered sequence $\{s_1, s_2, \cdots, s_m\} \subseteq \mathcal{I}$ such that
      $\forall x \in X\ \exists s_k$: $X_{s_k} = \{x\}$;
  (* Note that there is a unique s_k corresponding to each $x \in X$. *)
      Reindex elements of X as $\{x_1, x_2, \cdots, x_m\}$ such that for $1 \leq k \leq m$, $X_{s_k} = \{x_k\}$;
      $Y := \{x_k \mid 1 \leq k \leq m$ and k is odd $\}$;
      Find $k \in \{1, 2, \ldots, m\}$: k is odd and s_k is a false point of $\mathrm{OR}_\omega(Y)$
      or k is even and s_k is a true point of $\mathrm{OR}_\omega(Y)$;
      $\mathcal{I} := \mathcal{I} - \{s_k\}$;
  (* s_k is not the cut point of $\mathcal{I}$. *)

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17    until m ≤ N^{1/lε};
18    return(X);
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We first prove the correctness of procedure PRUNE. We do this by establishing the following three claims.

Claim 3.5.1 *If there is an $x \in X$ such that for every $p \in \mathcal{I}$, $x \in X_p$ implies $||X_p|| > 1$, then $x \in L$ implies $(X - \{x\}) \cap L \neq \emptyset$.*

Proof of Claim 3.5.1. Assume $x \in L$ satisfies the condition of the claim. Let $p \in \mathcal{I}$ be the cut point. It follows that p is a true point of x. Thus $x \in X_p$ which in turn implies $||X_p|| > 1$. Let $y \in X_p - \{x\}$. The cut point p is a true point of y which implies that $y \in L$. □

Claim 3.5.2 *In line 14 of* PRUNE, *there always exists an index k, $1 \leq k \leq m$, such that either k is odd and s_k is a false point of* $\mathrm{OR}_\omega(Y)$ *or k is even and s_k is a true point of* $\mathrm{OR}_\omega(Y)$. *Furthermore, s_k is not the cut point of $\mathcal{I}$.*

Proof of Claim 3.5.2.

Suppose s_k as claimed above exists. We prove that it cannot be the cut point of $\mathcal{I}$. If k is odd, clearly s_k is not the cut point because it is a false point of $\mathrm{OR}_\omega(Y)$ and a true point for $x_k \in Y$. If k is even, s_k cannot be the cut point because it is not a true point of any $y \in Y$ but it is a true point of $\mathrm{OR}_\omega(Y)$.

Now we prove that a string s_k as claimed does exist. Let $\mathcal{J} = \{s_i \mid 1 \leq i \leq m \text{ and } x_i \in Y\}$. From the bound on the size of the OR_ω function value it follows that $|\mathrm{OR}_\omega(Y)| = O((mN)^l)$. Let $g(\mathrm{OR}_\omega(Y)) = \{q_1, q_2, \cdots, q_t\}$ ordered by $\preceq$. Observe that $t = O((mN)^{l((1/l)-\epsilon)})$.

We consider two cases. In the first case, suppose there is a point $s_k \in \mathcal{J}$ such that s_k is a false point of $\mathrm{OR}_\omega(Y)$. Clearly s_k can be chosen as the required string.

Otherwise, it holds that every $s \in \mathcal{J}$ is a true point of $\mathrm{OR}_\omega(Y)$. In this case, we show that there is an even k such that s_k is a true point of $\mathrm{OR}_\omega(Y)$. Since $m = ||X|| > N^{1/l\epsilon}$, it holds that $t \leq O(m^{1-l^2\epsilon^2}) < \lfloor m/2 \rfloor = ||\mathcal{J}||$, for N greater than a fixed positive integer.

Now, order the strings in $\mathcal{J} \cup g(\mathrm{OR}_\omega(Y))$ by the p-selective ordering $\preceq$. Since $||\mathcal{J}|| > ||g(\mathrm{OR}_\omega(Y))||$, it follows by the pigeon-hole principle that there exists $j, 1 \leq j \leq t$, such that $q_j \preceq s_{k-1} \preceq s_{k+1} \preceq q_{j+1}$, for an even k, $1 \leq k \leq m$. Since s_{k-1} and s_{k+1} are in $\mathcal{J}$, they both are true points of $\mathrm{OR}_\omega(Y)$. Since $s_{k-1} \preceq s_k \preceq s_{k+1}$, it follows that s_k is also a true point of $\mathrm{OR}_\omega(Y)$.

This proves the claim. The index k is easy to compute, since checking if s_k is a true/false point for $\mathrm{OR}_\omega(Y)$ can be done in polynomial time. □

Claim 3.5.3 *Let N be an upper bound on the lengths of strings in X. Then* PRUNE(X) *runs in time bounded by a polynomial in $N \cdot ||X||$.*

Proof of Claim 3.5.3. To see that PRUNE(X) terminates, first observe that there are two repeat loops in the procedure PRUNE. Each time the inner repeat loop is entered the number of times it is executed is clearly bounded by $||X||$. Once the outer repeat loop is entered each time it loops results in a string s_k getting removed from $\mathcal{I}$ in *line* 16. Thus, each time this repeat loop is executed, $||\mathcal{I}||$ decreases by 1. Furthermore, at the end of every execution of the inner repeat loop, it holds that $||X|| \leq ||\mathcal{I}||$. Therefore, when the cardinality of $\mathcal{I}$ decreases the cardinality of X must also eventually decrease and finally get bounded by $N^{1/l\epsilon}$. Thus the outer repeat loop terminates implying that PRUNE terminates.

The number of executions of the outer loop is clearly bounded by the initial cardinality of $\mathcal{I}$, which in turn is bounded by $N^{(1/l)-\epsilon} \cdot ||X||$. Thus the total number of executions of both the repeat loops is polynomially bounded in N and $||X||$.

It is easy to see that every individual step in the procedure can be carried out in time bounded by a polynomial in N and $||X||$. In particular, we note that the detection of the index k in *line* 14 can also be carried out in time bounded by a polynomial in N and $||X||$, following the method indicated in Claim 3.5.2. □

Now, consider the procedure call DECIDE(x). Recall that $p(|x|)$ bounds the lengths of all strings and the number of children of any string in the self-reduction tree rooted at x. Observe that for every call PRUNE(F) made by DECIDE(x), it holds that $N \leq p(|x|)$ and $||F|| \leq p(|x|)(p(|x|))^{1/l\epsilon}$. Furthermore, it holds that $x \in L$ iff $F \cap L \neq \emptyset$. It is easy to see that all other steps of DECIDE can be executed in polynomial time. It follows that DECIDE is correct and the overall running time of procedure DECIDE(x) is bounded by a polynomial in $|x|$. ■

A set L is said to have OR_2 if there is a polynomial-time function OR_2 satisfying the following condition: $\mathrm{OR}_2(x, y) \in L$ iff $x \in L$ or $y \in L$. As a corollary to the proof of Theorem 3.5, we have the following result for bounded truth-table reduction.

Corollary 3.6 *Let L be a disjunctive self-reducible set such that L has OR_2. If $L \leq^p_{btt} A$ for a p-selective set A then $L \in \mathrm{P}$.*

Since SAT is a disjunctive self-reducible NP-complete set and has OR_ω with the property that $|\mathrm{OR}_\omega(X)| = O((\sum_{x\in X} |x|))$, for all finite subsets X of Σ^*. the next corollary directly follows.

Corollary 3.7 *If every set in* NP *is quasi-linear truth-table reducible to a p-selective set then* NP = P.

Another corollary follows from the fact that GI (the set of pairs of isomorphic labeled graphs) is also disjunctive self-reducible and has OR_ω [LT92, Ch89]. The OR_ω function for GI satisfies $|\mathrm{OR}_\omega(X)| = O(\sum_{x\in X} |x|^2)$. Hence we get the following corollary.

Corollary 3.8 *If* GI *is $O(n^{1/2-\epsilon})$-truth-table reducible to a p-selective set for $1 > \epsilon > 0$, then* GI $\in$ P.

Corollary 3.9 *For a class $\mathcal{K} \in \{\mathrm{PP}, \mathrm{C_=P}\}$, if every set in $\mathcal{K}$ is quasi-linear truth-table reducible to a p-selective set then $\mathcal{K} = \mathrm{P}$.*

Proof. Let $\mathcal{K} \in \{\mathrm{PP}, \mathrm{C_=P}\}$. Suppose every set in $\mathcal{K}$ is quasi-linear truth-table reducible to a p-selective set. Since p-selective sets are in P/poly [Sel79], it follows that $\mathcal{K} \subseteq$ P/poly. Both PP and $\mathrm{C_=P}$ have many-one complete sets that are one-word-decreasing self-reducible [OL91]. Since one-word-decreasing self-reducible sets in P/poly also belong to Σ^p_2 [Bal90], it follows that $\mathcal{K} \subseteq \Sigma^p_2$. Furthermore, if every set from $\mathcal{K} \in \{\mathrm{PP}, \mathrm{C_=P}\}$ is quasi-linear truth-table reducible to a p-selective set then, since NP $\subseteq$ PP and co-NP $\subseteq \mathrm{C_=P}$, and since p-selective sets are closed under complement, it follows from Corollary 3.7 that P = NP. Therefore, it follows that $\mathcal{K} = \mathrm{P}$. ■

With a pruning strategy as in Theorem 3.5 we can show that if $\mathrm{Mod}_k\mathrm{P}$ is $o(\log n)$-truth-table reducible to a p-selective set then $\mathrm{Mod}_k\mathrm{P} = \mathrm{P}$ [AA94]. However, in [O94] the same collapse result for $\mathrm{Mod}_k\mathrm{P}$ is proved for quasi-linear truth-table reductions to p-selective sets.

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