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Integer domination of Cartesian product graphs

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ABSTRACT

Given a graph G, a dominating set D is a set of vertices such that any vertex not in D has at least one neighbor in D. A $\{k\}$-dominating multiset D_k is a multiset of vertices such that any vertex in G has at least k vertices from its closed neighborhood in D_k when counted with multiplicity. In this paper, we utilize the approach developed by Clark and Suen (2000) to prove a "Vizing-like" inequality on minimum $\{k\}$-dominating multisets of graphs G, H and the Cartesian product graph $G\Box H$. Specifically, denoting the size of a minimum $\{k\}$-dominating multiset as $\gamma_{\{k\}}(G)$, we demonstrate that $\gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq 2k\,\gamma_{\{k\}}(G\Box H)$.

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1. Introduction

Let G be a simple undirected graph $G = (V, E)$ with vertex set V and edge set E. The open neighborhood of a vertex $v \in V(G)$ is denoted by $N_G(v)$, and the closed neighborhood of v is denoted by $N_G[v]$. A dominating set D of a graph G is a subset of $V(G)$ such that for all $v \in V(G)$, $N_G[v] \cap D \neq \emptyset$, and the size of a minimum dominating set is denoted by $\gamma(G)$. The Cartesian product of two graphs G and H, denoted $G\Box H$, is the graph with vertex set $V(G) \times V(H)$, where vertices $gh, g'h' \in V(G\Box H)$ are adjacent whenever $g = g'$ and $(h, h') \in E(H)$, or $h = h'$ and $(g, g') \in E(G)$.

In 1963, and again more formally in 1968, Vizing proposed a simple and elegant conjecture that has subsequently become one of the most famous open questions in domination theory.

Conjecture (*Vizing [11], 1968*). *Given graphs G and H, $\gamma(G)\gamma(H) \leq \gamma(G\Box H)$.*

Over the past forty years (see [1] and references therein), Vizing's conjecture has been shown to hold on certain restricted classes of graphs, and furthermore, upper and lower bounds on the inequality have been gradually tightened. Additionally, as numerous direct attempts on the conjecture have failed, research approaches have expanded to include explorations of similar inequalities for total, paired, and fractional domination [6]. However, the most significant breakthrough occurred in 2000, when Clark and Suen [4] demonstrated that $\gamma(G)\gamma(H) \leq 2\gamma(G\Box H)$. This "Vizing-like" inequality immediately suggested similar inequalities for total [8] and paired [9] domination (2008 and 2010, respectively). In 2011, we [3] improved the inequalities from [8,9] for total and paired domination by applying techniques similar to those of Clark and Suen, and also specific properties of binary matrices. In this paper, we explore *integer domination* (or $\{k\}$-domination), and again generate an improved inequality with this combined technique.

A *multiset* is a set in which elements are allowed to appear more than once, e.g. $\{1, 2, 2\}$. All graphs and multisets in this paper are finite. A $\{k\}$-dominating multiset D_k of a graph G is a multiset of vertices of $V(G)$ such that, for each $v \in V(G)$,

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the number of vertices of $N_G[v]$ contained in D_k (counted with multiplicity) is at least k. A $\gamma_{\{k\}}$-set of G is a minimum $\{k\}$-dominating multiset, and the size of a minimum $\{k\}$-dominating multiset is denoted by $\gamma_{\{k\}}(G)$. Additionally, note that a $\{1\}$-dominating multiset is equivalent to the standard dominating set.

The notion of a $\{k\}$-dominating multiset is equivalent to the more familiar notion of a $\{k\}$-dominating function. The study of $\{k\}$-dominating functions was first introduced by Domke, Hedetniemi, Laskar, and Fricke [5] (see also [7], pg. 90), and further explored by Brešar, Henning and Klavžar in [2]. The authors of [10] investigate integer domination in terms of graphs with specific packing numbers, and the authors of [2] prove the following "Vizing"-like inequality:

Theorem 1 *([2]). Given graphs G and H,* $\gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq k(k+1)\gamma_{\{k\}}(G\Box H)$.

Observe that for $k = 1$, Theorem 1 is equivalent to the bound proven by Clark and Suen. In this paper, we improve this upper bound from $O(k^2)$ to $O(k)$, and prove the following theorem:

Theorem 2. *Given graphs G and H,* $\gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq 2k\, \gamma_{\{k\}}(G\Box H)$.

Again, for $k = 1$, Theorem 2 is equivalent to the bound proven by Clark and Suen.

In Section 2, we explain the basic notation and concepts required for the proof, and in Section 3 we present the actual proof.

2. Preliminaries

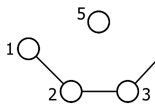
In this section, we introduce the necessary concepts and definitions used throughout the paper.

Given a universal set U, A is said to be a multiset of U if its elements are only those present in U. We denote the number of occurrences of a particular element x in A by $|A|_x$. The union of multisets is denoted by $\uplus$. Letting A and B be multisets of U, then $A \uplus B$ is a multiset of U such that for each x in U, $|A \uplus B|_x = |A|_x + |B|_x$. Similarly, $A \cap B$ is a multiset of U satisfying $|A \cap B|_x = min\{|A|_x, |B|_x\}$. The union of a multiset A with itself t times is denoted by $\uplus^t A$. A multiset B is a submultiset of A if for each x, $|B|_x \leq |A|_x$. The cardinality of a multiset A is the summation over the number of occurrences of each element in it, i.e. $|A| = \sum_{x\in U} |A|_x$. Finally, for multiset A and a set S subset of U, $|A|_S$ is defined as $\sum_{x\in S} |A|_x$.

Now consider a graph G. Let $P^G = \{P_1, P_2, \ldots, P_t\}$ be a multiset whose elements are subsets of $V(G)$. Then P^G is called a *k-partition* of $V(G)$ if each vertex x in G is present in exactly k of the sets $P_1, \ldots, P_t$. We will now see that any $\gamma_{\{k\}}$-set of G (and a specific assignment of vertices to dominators), naturally induces a k-partition on $V(G)$. Let $\{u_1, \ldots, u_{\gamma_{\{k\}}}\}$ be a minimum $\{k\}$-dominating multiset of G. To each dominator u_i, we associate a subset P_i^G of $V(G)$ as follows. Recall that for each vertex x in G there exists at least k dominators (say $u_{j_1}, \ldots, u_{j_k}$) in $N_G[x]$. Therefore, by only including x in the sets $P_{j_1}^G, \ldots, P_{j_k}^G$, the multiset $\{P_1^G, \ldots, P_{\gamma_{\{k\}}}^G\}$ is a k-partition of $V(G)$. Additionally, note that for each i, the set P_i^G is a subset of $N_G[u_i]$.

Given a graph G, we will now define the concept of domination among multisets of $V(G)$. Given a vertex x, a vertex y is said to be a *dominator* of x if $y \in N_G[x]$. Let A, B be multisets of $V(G)$. We say that A *dominates* B if for each $x \in B$, $|A|_{N_G[x]} \geq |B|_x$. In other words, the number of dominators of x in A (when counted with multiplicity) is at least the number of occurrences of x in B. It is important to note that a multiset D is a $\{k\}$-dominating set for G if and only if D dominates $\uplus^k V(G)$.

Example 1. Consider the following graph G, and let $A = \{1, 2, 3, 4, 5, 5\}$ be a minimum 2-dominating multiset of G. Assuming that vertex 1 is assigned to be dominated by vertices 1 and 2 (denoted as $1 \to \{1, 2\}$), vertex $2 \to \{2, 3\}$, vertex $3 \to \{2, 3\}$, vertex $4 \to \{3, 4\}$ and vertex $5 \to \{5, 5\}$.



Then $P^G = \big\{\{1\}, \{1, 2, 3\}, \{2, 3, 4\}, \{4\}, \{5\}, \{5\}\big\} = \{P_1^G, \ldots, P_6^G\}$ is a 2-partition of $V(G)$ induced by A. Observe that each $v \in V(G)$ appears in exactly two sets in P^G, and that each P_i^G is a set (i.e., it contains no duplicated elements), but P^G itself is a multiset.

We also observe that the multiset $A = \{1, 2, 2\}$ dominates the multiset $B = \{1, 1, 1, 2, 3\}$, since $|A|_{N_G[x]} \geq |B|_x$ for all $x \in B$. $\Box$

We now give definitions associated with the product graph $G\Box H$. For any $h \in H$, the subgraph of $G\Box H$ induced by the set $V(G) \times \{h\}$ is said to be a *G-fiber*. The *G-neighborhood* of vertex gh consists of the neighbors of gh which lie in the fiber $V(G) \times \{h\}$, including itself. Similarly, H-fibers and the H-neighborhood can be defined. Note that for any vertex gh, the intersection of its G-neighborhood and H-neighborhood is the singleton set $\{gh\}$. Now let A be a multiset of $V(G\Box H)$. The projection of A on G is a multiset of $V(G)$ defined as follows.

$$\Phi_G(A) = \left\{g \in V(G)\ ,\ \text{with } \left|\Phi_G(A)\right|_g = \sum_{h\in V(H)} |A|_{gh}\right\}.$$

Similarly $\Phi_H(A)$ can be defined.

We end this section by stating a proposition whose proof is a straightforward application of the definitions stated above.

Proposition 1. *Given graphs G, H*

1. *Let A_1, B_1, A_2, B_2 be multisets of $V(G)$. If A_1 dominates B_1, and A_2 dominates B_2, then $A_1 \uplus A_2$ dominates $B_1 \uplus B_2$.*
2. *Let A be multiset of $V(G\Box H)$. Then $|\Phi_G(A)| = |\Phi_H(A)| = |A|$.*

3. Main proof

We now start with the details of our proof. Let $\{u_1, \ldots, u_{\gamma_{\{k\}}(G)}\}$ and $\{\overline{u}_1, \ldots, \overline{u}_{\gamma_{\{k\}}(H)}\}$ be minimum $\{k\}$-dominating multisets of G, H, respectively, and let $I = \{1, \ldots, \gamma_{\{k\}}(G)\}$ and $J = \{1, \ldots, \gamma_{\{k\}}(H)\}$ be the corresponding sets of indices. Additionally, let $P^G = \{P^G_1, \ldots, P^G_{\gamma_{\{k\}}(G)}\}$ and $P^H = \{P^H_1, \ldots, P^H_{\gamma_{\{k\}}(H)}\}$ be the induced k-partitions of $V(G)$ and $V(H)$, respectively. Finally, let D be a minimum $\{k\}$-dominating multiset for graph $G\Box H$.

Proposition 2. *Let $T \subseteq I$, $A = \uplus_{i\in T}\{u_i\}$, and $C = \uplus_{i\in T} P^G_i$. Then A dominates C. Furthermore, for any other multiset B of $V(G)$, if B dominates C, then $|B| \geq |T|$.*

Proof. We first prove A dominates C. Since $P^G_i \subseteq N_G[u_i]$, u_i dominates P^G_i. Therefore, $\uplus_{i\in T}\{u_i\}$ dominates $\uplus_{i\in T} P^G_i$, i.e. A dominates C. Now let multiset $W = \uplus_{i\notin T}\{u_i\}$. Since B is any multiset dominating C, by Proposition 1.1 we have $B \uplus W$ dominates $\left(\left(\uplus_{i\in T} P^G_i\right) \uplus \left(\uplus_{i\notin T} P^G_i\right)\right)$. Now as $B \uplus W$ dominates $\uplus^k V(G) = \uplus_{i\in I} P^G_i$, it is a $\{k\}$-dominating multiset for graph G. Finally, as $A \uplus W = \{u_1, \ldots, u_{\gamma_k(G)}\}$ is a $\gamma_{\{k\}}$-set of G, $|B \uplus W| \geq |A \uplus W|$. Therefore, $|B| \geq |A| = |T|$. □

Now observe that since P^G and P^H are k-partitions of $V(G), V(H)$, respectively, $P^G \times P^H$ is a k^2-partition of $V(G\Box H)$. Let $\overline{V} = \uplus^{k^2} V(G\Box H) = \uplus_{i\in I} \uplus_{j\in J}(P^G_i \times P^H_j)$. We refer to $P^G_i \times P^H_j$ as a *block* in $V(G\Box H)$. Thus, in total, there are $\gamma_{\{k\}}(G) \times \gamma_{\{k\}}(H)$ blocks (when counted with multiplicity). Furthermore, each vertex gh in $G\Box H$ appears in exactly k^2 of the blocks in $P^G \times P^H$. Finally, as D dominates $\uplus^k V(G\Box H)$, we have $\uplus^k D$ dominates $\overline{V}$.

We now define a notion of "strips" in $G\Box H$, which is similar to the notion of fibers defined earlier. For $j \in J$, let each of the sets $V(G) \times P^H_j$ be a G-strip, and similarly the sets $P^G_i \times V(H)$ be H-strips. Note that the G-strips (H-strips) form a k-partition of $V(G\Box H)$. Also, each G-strip contains $\gamma_k(G)$ blocks, and each H-strip contains $\gamma_k(H)$ blocks.

We now assign a dominator from D to each copy of vertex gh in $\overline{V}$. As each gh appears in exactly k^2 blocks in $P^G \times P^H$, it is sufficient to assign a dominator to gh corresponding to each of the blocks containing it. Let F be this assignment function. Thus, F is defined over the set $V(G\Box H) \times (P^G \times P^H)$, and $F(gh, P^G_i \times P^H_j)$ is an element of D whenever gh appears in block $P^G_i \times P^H_j$ and is $\emptyset$ (a nil value) otherwise.

Proposition 3. *There exists an assignment F such that, for any vertex gh, the dominators assigned to it in each strip (G-strip or H-strip) are a subset of D. In other words, for each $i \in I$, $\uplus_{j\in J} F(gh, P^G_i \times P^H_j)$ is a subset of D, and for each $j \in J$, $\uplus_{i\in I} F(gh, P^G_i \times P^H_j)$ is a subset of D.*

Proof. Consider a vertex $gh \in V(G\Box H)$. Let $d_0, \ldots, d_{k-1}$ be the k (not necessarily distinct) dominators of gh in D. Let $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$ be indices in I, J, respectively, such that for $1 \leq r, s \leq k$, the block $P^G_{i_r} \times P^H_{j_s}$ contains vertex gh. Now define $F(gh, P^G_{i_r} \times P^H_{j_s})$ as $d_{(r+s) mod\ k}$. Recall $F(gh, P^G_i \times P^H_j)$ is defined as $\emptyset$ if $gh \notin P^G_i \times P^H_j$. Thus for any index i_r, $\uplus_{j\in J} F(gh, P^G_{i_r} \times P^H_j) = \{d_{(r+1) mod\ k}, \ldots, d_{(r+k) mod\ k}\} = \{d_0, \ldots, d_{k-1}\}$. Hence for any index $i \in I$ we have $\uplus_{j\in J} F(gh, P^G_i \times P^H_j)$ is equal to $\{d_0, \ldots, d_{k-1}\}$ if $g \in P^G_i$, and empty otherwise. This proves the first part. The proof for the second part similarly follows. □

Given $gh \in P^G_i \times P^H_j$, we define $F(gh, P^G_i \times P^H_j)$ in the same way as in proof of Proposition 3. We now define a notion of H-dominated and G-dominated blocks in $P^G \times P^H$. A block $P^G_i \times P^H_j$ is said to be *H-dominated* if for each $h \in P^H_j$, there exists a $g \in P^G_i$ such that $F(gh, P^G_i \times P^H_j)$ belongs in the H-neighborhood of gh. Recall the H-neighborhood of gh consists of neighbors of gh (including itself) in the fiber $g \times V(H)$. A G-dominated block is defined similarly.

Let N_i be the number of blocks in strip $P^G_i \times V(H)$ which are H-dominated, and $\overline{N_j}$ be the number of blocks in strip $V(G) \times P^H_j$ which are G-dominated.

Proposition 4. $\sum_{i\in I} N_i + \sum_{j\in J} \overline{N_j} \geq \gamma_k(G)\gamma_k(H)$.

Proof. We first show that each block is G-dominated, H-dominated or both. Consider a block $P^G_i \times P^H_j$ which is not G-dominated. Then there exists a vertex g_0 in P^G_i such that for each $h \in P^H_j$, $F(g_0h, P^G_i \times P^H_j)$ does not lie in the G-neighborhood of g_0h. Suppose $P^G_i \times P^H_j$ is also not H-dominated. Then there will exist a vertex h_0 in P^H_j such that for each $g \in P^G_i$, $F(gh_0, P^G_i \times P^H_j)$ does not lie in the H-neighborhood of gh_0. But this means that $F(g_0h_0, P^G_i \times P^H_j)$ lies neither in the G-neighborhood nor H-neighborhood of g_0h_0. This is a contradiction. Recall that $\sum_{i\in I} N_i$ and $\sum_{j\in J} \overline{N_j}$ are the total number of G-dominated and H-dominated blocks, respectively. As there are $\gamma_k(G)\gamma_k(H)$ blocks, we see that $\sum_{i\in I} N_i + \sum_{j\in J} \overline{N_j} \geq \gamma_k(G)\gamma_k(H)$. □

Now we give an upper bound on the total number of G-dominated (and H-dominated) blocks.

Proposition 5. $\sum_{i\in I} N_i,\ \sum_{j\in J} \overline{N_j} \leq k\gamma_k(G\Box H)$.

Proof. We prove that $\sum_{i\in I} N_i \le k\gamma_k(G\Box H)$. The proof for $\sum_{j\in J} \overline{N_j} \le k\gamma_k(G\Box H)$ follows similarly. For $i \in I$, define

$$Y_i = \uplus \{P_j^H \mid P_i^G \times P_j^H \text{ is } H\text{-dominated}\}$$
$$S_i = D \cap \left(\uplus^k \left(P_i^G \times V(H)\right)\right).$$

We now divide the proof in three parts.

Claim 1. *For each $i \in I$, $\Phi_H(S_i)$ dominates Y_i.*

Proof. For any fixed $i \in I$ consider a vertex $h \in Y_i$. Let it have α occurrences in Y_i. This means there are α blocks $P_i^G \times P_{j_1}^H, \dots, P_i^G \times P_{j_\alpha}^H$ which are H-dominated and h belongs in each of $P_{j_t}^H$, $1 \le t \le \alpha$. Therefore, for each t, there exists a vertex $g_t h \in P_i^G \times P_{j_t}^H$ such that $F(g_t h, P_i^G \times P_{j_t}^H)$ belongs in H-neighborhood of $g_t h$. Let this dominator $F(g_t h, P_i^G \times P_{j_t}^H)$ be $g_t h_t$. Then, $h_1, \dots, h_\alpha$ must lie in $N_H[h]$. We will show that $\{g_1 h_1, \dots, g_\alpha h_\alpha\}$ is a subset of D.

Let β be the number of distinct elements in $\{g_1 h, \dots, g_\alpha h_\alpha\}$ (i.e. when counted without multiplicity). Let $L_1, \dots, L_\beta$ be a partition of $[\alpha]$ such that for t, t' lying in same L_r, $g_t h = g_{t'} h$ and for t, t' lying in different L_r's $g_t h \ne g_{t'} h$. Now from Proposition 3 we have that $\uplus_{j\in J} F(gh, P_i^G \times P_j^H)$ is a subset of D. Thus, for each r, $\uplus_{t\in L_r} g_t h_t$ is a subset of D. Also for $r \ne s$, the intersection of $\uplus_{t\in L_r} g_t h_t$ and $\uplus_{t'\in L_s} g_{t'} h_{t'}$ is empty. This is because if $g_t \ne g_{t'}$ then $g_t h_t$ and $g_{t'} h_{t'}$ will belong in different G-fibers. Hence, $\uplus_{t\in[\alpha]} g_t h_t$ is a subset of D.

Now note that since $\{g_1 h, \dots, g_\alpha h_\alpha\}$ is a subset of D and each $g_t h_t$ lies in strip $P_i^G \times V(H)$, we have that $\{h_1, \dots, h_\alpha\}$ is a subset of $\Phi_H(S_i)$. Hence, there exist at least α dominators for vertex h in $\Phi_H(S_i)$ (when counted with multiplicity). This proves $\Phi_H(S_i)$ dominates Y_i. □

Claim 2. *For each $i \in I$, $N_i \le |\Phi_H(S_i)|$.*

Proof. Let $T_i = \{j \mid P_i^G \times P_j^H \text{ is } H\text{-dominated}\}$. Then $Y_i = \uplus_{j\in T_i} P_j^H$. Now since $\Phi_H(S_i)$ dominates Y_i from Proposition 2 we have that $|\Phi_H(S_i)| \ge |T_i| = N_i$. □

Claim 3. *The multiset $\uplus_{i\in I} S_i$ is equal to $\uplus^k D$.*

Proof. Consider any $gh \in D$. Let n_0 be the number of occurrences of gh in D. Now consider any $P_i^G \in P^G$. If P_i^G contains g, then $\uplus^k\left(P_i^G \times V(H)\right)$ will contain k copies of gh, and S_i will contain n_0 copies of gh. If $g \notin P_i^G$, then S_i will not contain zero copies of gh. Hence there are exactly k S_i's which have n_0 copies of gh, and all the remaining do not contain vertex gh. Additionally, since $\uplus_{i\in I} S_i$ contains kn_0 copies of gh we see $\uplus_{i\in I} S_i = \uplus^k D$. □

Finally, $\sum_{i\in I} N_i \le \sum_{i\in I} |\Phi_H(S_i)| = |\uplus_{i\in I} \Phi_H(S_i)| = |\uplus^k D| = k\gamma_k(H)$. Thus the result follows. □

Finally from Propositions 4 and 5 we get $\gamma_k(G)\gamma_k(H) \le 2k\gamma_k(G\Box H)$.

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