CS738: Advanced Compiler Optimizations

Simply Typed Lambda Calculus

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Reference Book

Types and Programming Languages by Benjamin C. Pierce

Simple Types over Bool

$T \quad :=$ – Types

Simple Types over Bool

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		Bool	– Boolean Type

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type constructor $\rightarrow$ is right-associative, i.e., $T_1 \rightarrow T_2 \rightarrow T_3$ stands for $T_1 \rightarrow (T_2 \rightarrow T_3)$

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Simply Typed λ-terms with conditions and Booleans

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	|	`if` t `then` t `else` t	– *conditional*

Recap: The Set of Values

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		$\lambda x : T.\,\mathrm{t}$	*– Abstraction Value*

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$$(\lambda x : T_1.\, t_1)v_2 \rightarrow [x \mapsto v_2]t_1 \qquad \text{(E-APPABS)}$$

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- $\Gamma, x : T$ denotes extending Γ with a new variable x having type T
  - The name x is assumed to be distinct from any existing names in Γ

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$$\frac{\Gamma, x : T_1 \vdash \mathsf{t}_2 : T_2}{\Gamma \vdash \lambda x : T_1 .\, \mathsf{t}_2 : T_1 \to T_2} \qquad \text{(T-ABS)}$$

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- If $\Gamma \vdash \mathrm{t}_1\ \mathrm{t}_2 : R$, then $\exists T_1\ s.t.\ \Gamma \vdash \mathrm{t}_1 : T_1 \to R$ and $\Gamma \vdash \mathrm{t}_2 : T_1$.

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- If t is typeable, then its type is unique.
- Moreover, there is just one derivation of this typing built from the inference rules.

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- **Permutation:** If $\Gamma \vdash \mathrm{t} : T$ and Δ is a permutation of Γ, then $\Delta \vdash \mathrm{t} : T$.

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  - If $\Gamma \vdash \mathrm{t} : T$ and $\mathrm{t} \rightarrow \mathrm{t}'$, then $\Gamma \vdash \mathrm{t}' : T$.