CS618: Program Analysis
2016-17 I^{st} Semester

Interprocedural Data Flow Analysis

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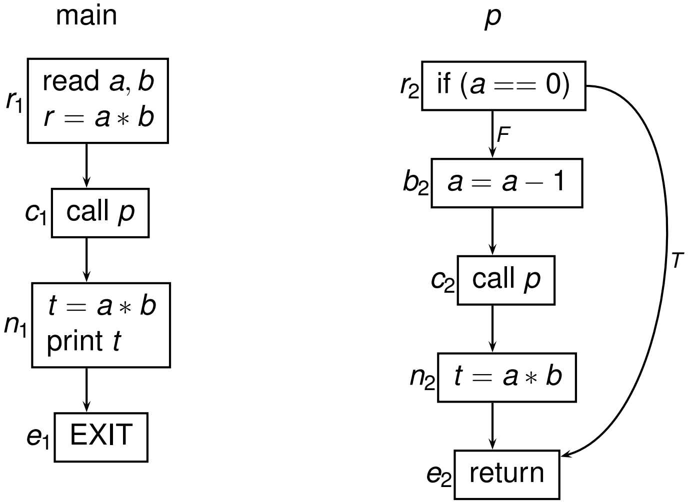
Interprocedural Analysis: WHY?

Is $a * b$ available at IN of n_1?



Challenges

- Infeasible paths
- Recursion
- Function pointers and virtual functions
- Dynamic functions (functional programs)

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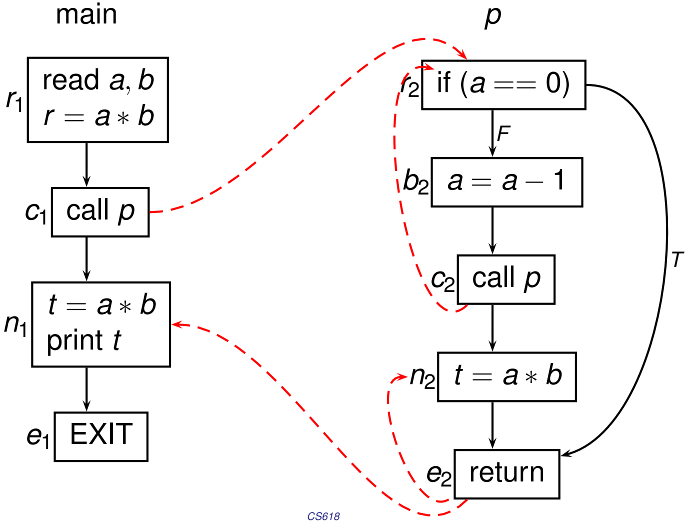
Infeasible Paths

How to avoid data flowing along invalid paths?

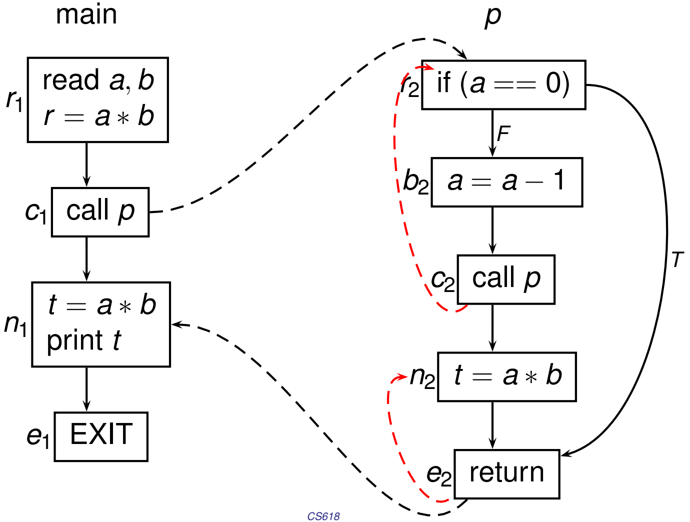
$$r_1 \rightarrow c_1 \rightarrow r_2 \rightarrow b_2 \rightarrow c_2 \rightarrow r_2 \rightarrow e_2 \rightarrow n_1$$



Recursion

How to handle Infinite paths?

$\ldots \rightarrow r_2 \rightarrow c_2 \rightarrow r_2 \rightarrow c_2 \rightarrow r_2 \ldots$



Function Variables

- Target of a function can not be determined statically
- Function Pointers (including virtual functions)

```
double (*fun)(double arg);
...
if (cond)
    fun = sqrt;
else
    fun = fabs;
...
fun(x);
```

- Dynamically created functions (in functional languages)
- No static control flow graph!

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Two Approaches

- Functional approach
  - procedures as structured blocks
  - input-output relation (*functions*) for each block
  - *function* used at call site to compute the effect of procedure on program state
- Call-strings approach

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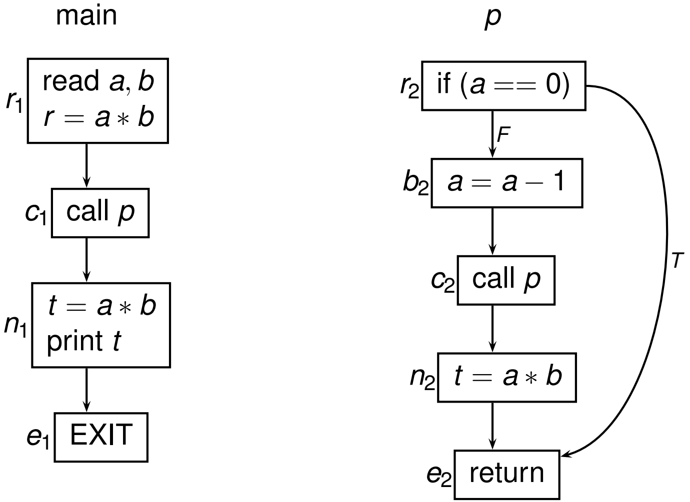
Notations and Terminology

Control Flow Graph

One per procedure



Control Flow Graph for Procedure p

- Single instruction basic blocks
- Unique exit block, denoted e_p
- Unique entry block, denoted r_p (root block)
- Edge (m, n) if direct control transfer from (the end of) block m to (the start of) block n
- Path: $(n_1, n_2, \ldots, n_k)$

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- Parameterless procedures, to ignore the problems of
  - *aliasing*
  - recursion stack for formal parameters
- No procedure variables (pointers, virtual functions etc.)

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Data Flow Framework

- (L, F): data flow framework
- L: a meet-semilattice
  - Largest element Ω
- F: space of propagation functions
- $f_{(m,n)} \in F$ represents propagation function for edge (m, n) of control flow graph $G = (N, E)$

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  - Change of DF values from the *start* of m, through m, to the *start* of n

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Data Flow Equations

$$
\begin{aligned}
x_r &= \mathit{BoundaryInfo} \\
x_n &= \bigwedge_{(m,n)\in E} f_{(m,n)}(x_m) \qquad n \in N - r
\end{aligned}
$$

- MFP solution, approximation of MOP

$$
y_n = \bigwedge \{ f_p(\mathit{BoundaryInfo}) : p \in \mathrm{path}_G(r, n) \} \quad n \in N
$$

Functional Approach
to
Interprocedural Analysis

Functional Approach

- Procedures treated as structures of blocks
- Computes relationship between DF value at entry node and related data at *any* internal node of procedure
- At call site, DF value propagated directly using the computed relation

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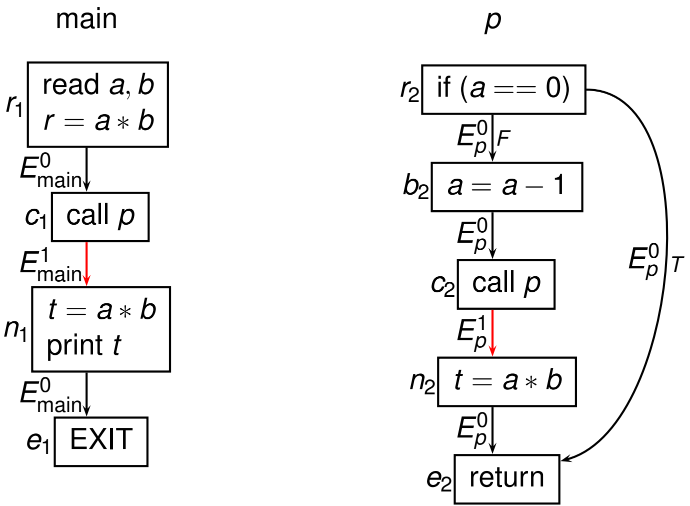
Interprocedural Flow Graph

First Representation:

$$
\begin{aligned}
G &= \bigcup \{G_p : p \text{ is a procedure in program}\} \\
G_p &= (N_p, E_p, r_p) \\
N_p &= \text{set of all basic block of } p \\
r_p &= \text{root block of } p \\
E_p &= \text{set of edges of } p \\
&= E_p^0 \cup E_p^1 \\
(m, n) \in E_p^0 &\Leftrightarrow \text{direct control transfer from } m \text{ to } n \\
(m, n) \in E_p^1 &\Leftrightarrow m \text{ is a call block, and } n \text{ immediately follows } m
\end{aligned}
$$

Interprocedural Flow Graph: 1st Representation



Interprocedural Flow Graph

Second representation

$$
\begin{aligned}
G^* &= (N^*, E^*, r_1) \\
r_1 &= \text{root block of main} \\
N^* &= \bigcup_p N_p \\
E^* &= E^0 \cup E^1 \\
E^0 &= \bigcup_p E_p^0 \\
(m, n) \in E^1 &\Leftrightarrow (m, n) \text{ is either a } \textit{call} \text{ edge} \\
& \quad\ \text{or a } \textit{return} \text{ edge}
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Interprocedural Flow Graph

- Call edge (m, n):
  - m is a call block, say calling p
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- Call edge (m, r_p) *corresponds* to return edge (e_q, n)

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  - if $p = q$ and
  - $(m, n) \in E^1_s$ for some procedure s

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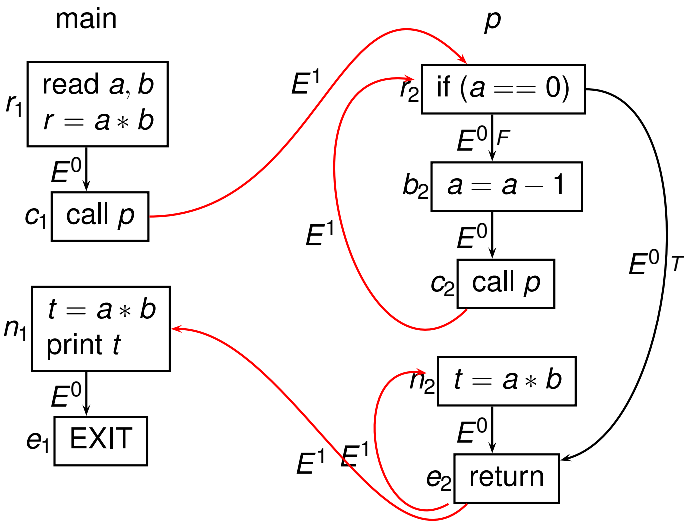
Interprocedural Flow Graph: 2nd Representation



Interprocedurally Valid Paths

- G^* ignores the special nature of call and return edges
- Not all paths in G^* are feasible
  - do not represent potentially valid execution paths
- IVP(r_1, n): set of all interprocedurally valid paths from r_1 to n
- Path $q \in \text{path}_{G^*}(r_1, n)$ is in IVP(r_1, n)

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Proper sequence

- q_1 without any return edge is proper
- let $q_1[i]$ be the first return edge in q_1. q_1 is proper if
  - $i > 1$; and
  - $q_1[i-1]$ is call edge corresponding to $q_1[i]$; and
  - q_1' obtained from deleting $q_1[i-1]$ and $q_1[i]$ from q_1 is proper

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Interprocedurally Valid Complete Paths

- $\text{IVP}_0(r_p, n)$ for procedure p and node $n \in N_p$
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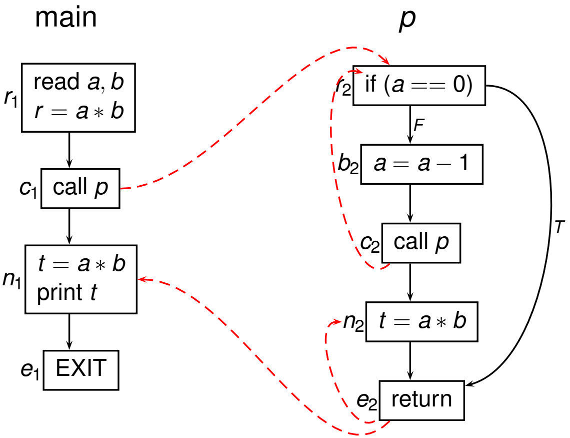
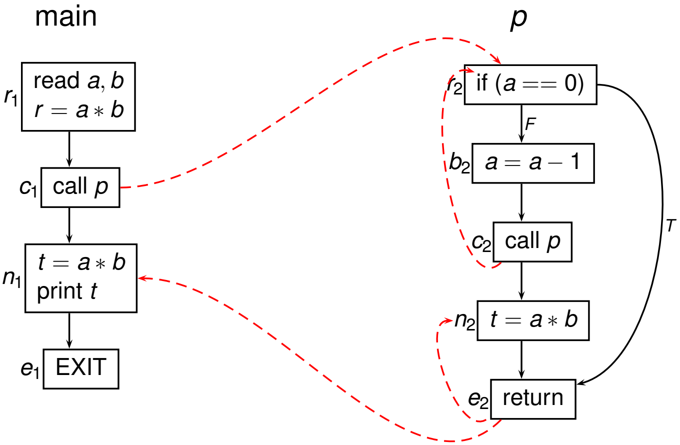
  - Each call edge has corresponding return edge in q restricted to E^1

IVPs

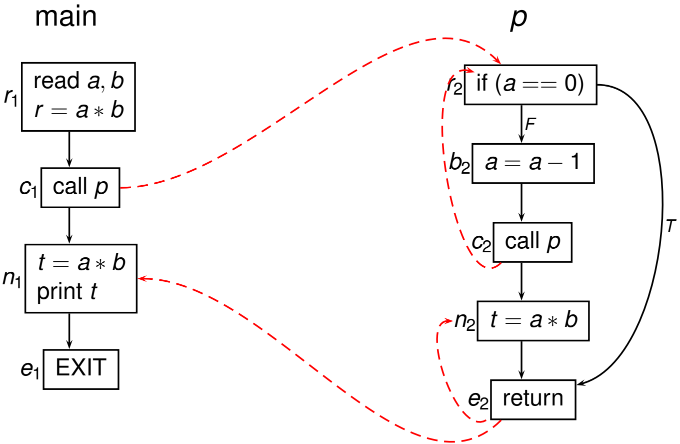

main
p
r_1
read a, b
$r = a * b$
c_1
call p
n_1
$t = a * b$
print t
e_1
EXIT
r_2
if $(a == 0)$
F
b_2
$a = a - 1$
c_2
call p
T
n_2
$t = a * b$
e_2
return

IVP*s*



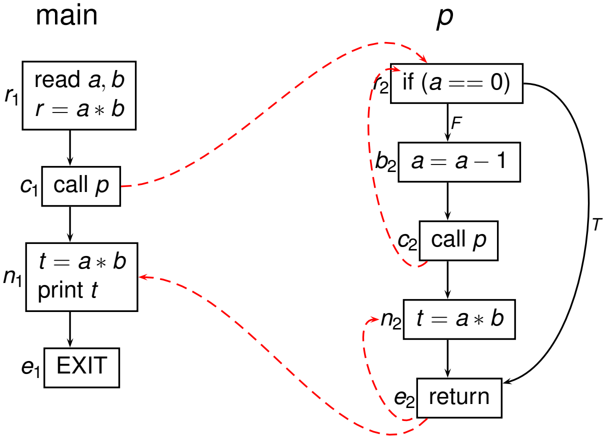
$$r_1 \rightarrow c_1 \rightarrow r_2 \rightarrow c_2 \rightarrow r_2 \rightarrow e_2 \rightarrow n_2 \rightarrow e_2 \rightarrow n_1 \rightarrow e_1 \in \mathrm{IVP}(r_1, e_1)$$

IVPs

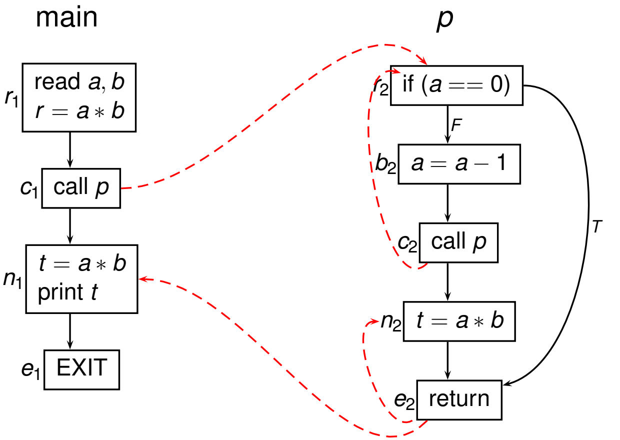


$$r_1 \rightarrow c_1 \rightarrow r_2 \rightarrow c_2 \rightarrow r_2 \rightarrow e_2 \rightarrow n_2 \rightarrow e_2 \rightarrow n_1 \rightarrow e_1 \in \mathrm{IVP}(r_1, e_1)$$

IVPs

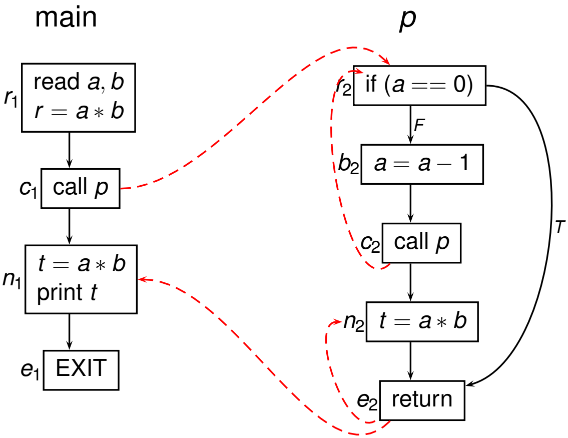


$$r_1 \to c_1 \to r_2 \to c_2 \to r_2 \to e_2 \to n_1 \to e_1 \notin \text{IVP}(r_1, e_1)$$



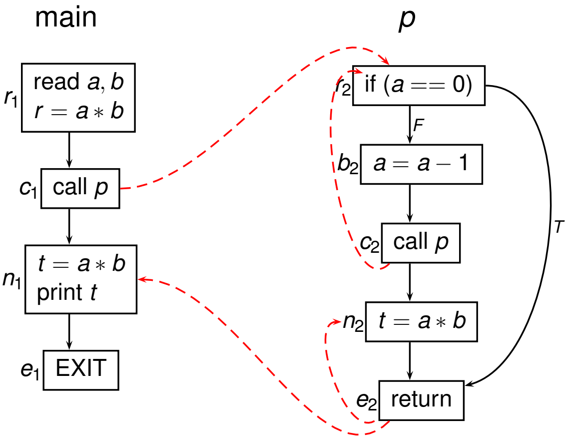
$$r_1 \rightarrow c_1 \rightarrow r_2 \rightarrow c_2 \rightarrow r_2 \rightarrow e_2 \rightarrow n_1 \rightarrow e_1 \notin \mathrm{IVP}(r_1, e_1)$$

IVPs



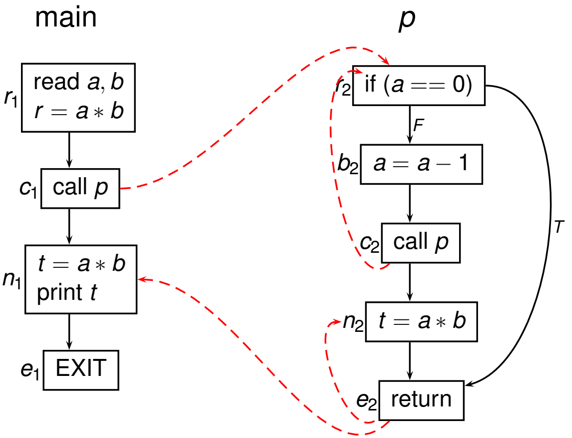
$$r_2 \rightarrow c_2 \rightarrow r_2 \rightarrow e_2 \rightarrow n_2 \in \mathrm{IVP}_0(r_2, n_2)$$

IVPs



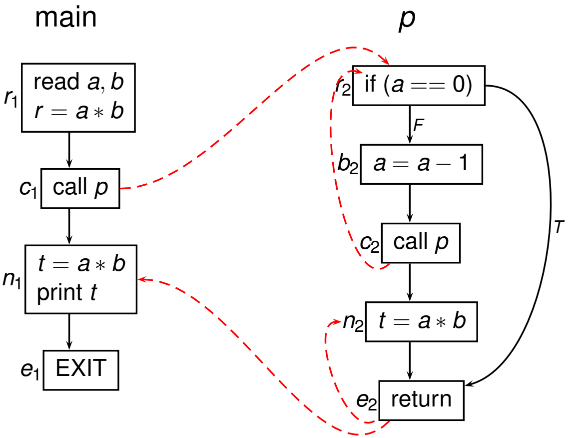
$$r_2 \rightarrow c_2 \rightarrow r_2 \rightarrow e_2 \rightarrow n_2 \in \text{IVP}_0(r_2, n_2)$$

IVPs



$$r_2 \rightarrow c_2 \rightarrow r_2 \rightarrow c_2 \rightarrow e_2 \rightarrow n_2 \notin \mathrm{IVP}_0(r_2, n_2)$$

IVPs



$$r_2 \rightarrow c_2 \rightarrow r_2 \rightarrow c_2 \rightarrow e_2 \rightarrow n_2 \notin \mathrm{IVP}_0(r_2, n_2)$$

Path Decomposition

$$
\begin{array}{rcl}
q & \in & \mathrm{IVP}(r_{\mathrm{main}}, n) \\
 & \Leftrightarrow & \\
q & = & q_1 \parallel (c_1, r_{p_2}) \parallel q_2 \parallel \cdots \parallel (c_{j-1}, r_{p_j}) \parallel q_j \\
 & & \text{where for each } i < j,\, q_i \in \mathrm{IVP}_0(r_{p_i}, c_i) \text{ and } q_j \in \mathrm{IVP}_0(r_{p_j}, n)
\end{array}
$$