Program Analysis

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Foundations of Data Flow Analysis

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Taxonomy of Dataflow Problems

- Categorized along several dimensions
  - the information they are designed to provide
  - the direction of flow
  - confluence operator
- Four kinds of dataflow problems, distinguished by
  - the operator used for confluence or divergence
  - data flows backward or forward

Taxonomy of Dataflow Problems

Confluence→ Direction↓	$\cup$	$\cap$
Forward	Reaching Definition	Available Expressions
Backward	Live Variables	Very Busy Expressions

When does Data Flow Analysis Works?

- Suitable initial values and boundary conditions
- Suitable domain of values
  - Bounded, Finite
- Suitable meet operator
- Suitable flow functions
  - monotonic, closed under composition
- But what is “SUITABLE” ?

Why Data Flow Analysis Works?

- Suitable initial values and boundary conditions
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- But what is “SUITABLE” ?

Lattice Theory

Partially Ordered Sets

- Posets
- S : a set
- $\leq$: a relation
- $(S, \leq)$ is a poset if $\forall\ x, y, z \in S$
  - $x \leq x$ (reflexive)
  - $x \leq y$ and $y \leq x \Rightarrow x = y$ (antisymmetric)
  - $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitive)

Chain

- Linear Ordering
- Poset where every pair of elements is comparable
- $x1 \leq x2 \leq \ldots \leq xk$ is a chain of length k
- We are interested in chains of **finite** length

Observation

- Any finite nonempty subset of a poset has minimal and maximal elements
- Any finite nonempty chain has unique minimum and maximum elements

Semilattice

- Set S and meet $\wedge$
- $\forall\ x, y, z \in S$
  - $x \wedge x = x$ (idempotent)
  - $x \wedge y = y \wedge x$ (commutative)
  - $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (associative)
- Partial order for semilattice
  - $x \leq y$ if and only if $x \wedge y = x$
  - Reflexive, antisymmetric, transitive

Border Element

- Top Element ($\top$)
  - $\forall x \in S, x \wedge \top = \top \wedge x = x$

- (Optional) Bot Element ($\bot$)
  - $\forall x \in S, x \wedge \bot = \bot \wedge x = \bot$

Familiar (semi)lattices

- Powerset for a set S, 2^S
- Meet $\wedge\ is\ \cap$
- Partial Order is $\subseteq$
- Top element is S
- Bottom element is $\emptyset$

Familiar (semi)lattices

- Powerset for a set S, 2^S
- Meet $\wedge\ is$ $\cup$
- Partial Order is $\supseteq$
- Top element is $\emptyset$
- Bottom element is S, the universal set

Greatest Lower bound

- glb of x and y is an element g s.t.
  - $g \leq x$
  - $g \leq y$
  - If $z \leq x$ and $z \leq y$ then $z \leq g$

- $x \wedge y$ is glb of x and y (Prove!)

Semi (?)-Lattice

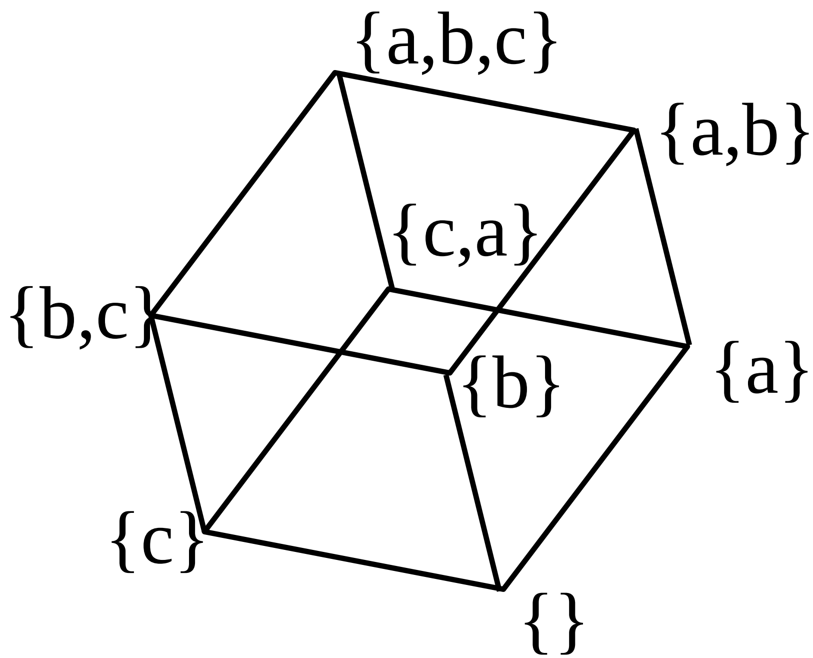
- We can define symmetric concepts:
  - $\geq$ order
  - $\vee$, Join operation
  - Least upper bound (lub)
- A complete lattice has both meet and join
  - Powerset lattice
- We will talk about “meet” semi-lattices only

Lattice Diagrams

- Graphical view of posets
- Elements = nodes in the graph
- If $x < y$ then x is depicted lower than y in the diagram
- An edge between x and y (x lower than y) implies $x < y$ and no other element z s.t. $x < z < y$ (i.e. transitivity excluded)

Lattice Diagram

$$\top = \{a, b, c\}$$
$$\bot = \emptyset$$
$$meet = \cap$$

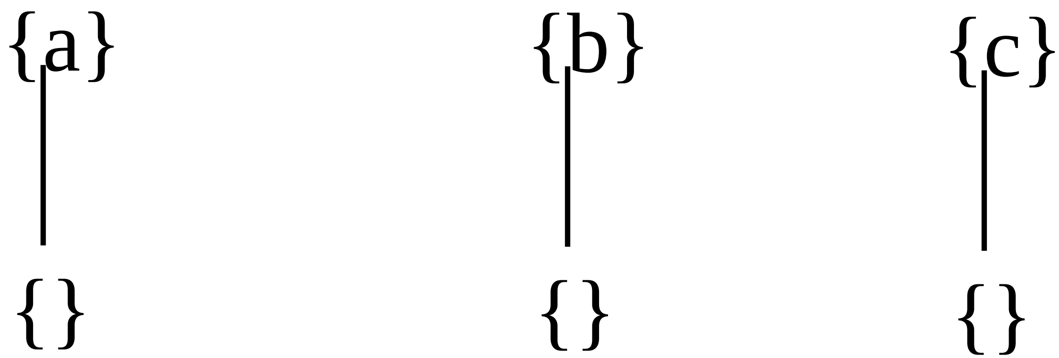


Lattice of superset relation

x ∧ y (glb): the highest *z* for which there are paths downward from both x and y.

What if we have a large number of elements?

- Combine simple lattices to build a complex one
- Superset lattices for singletons

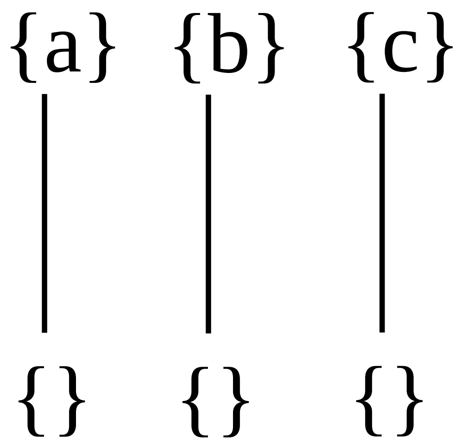


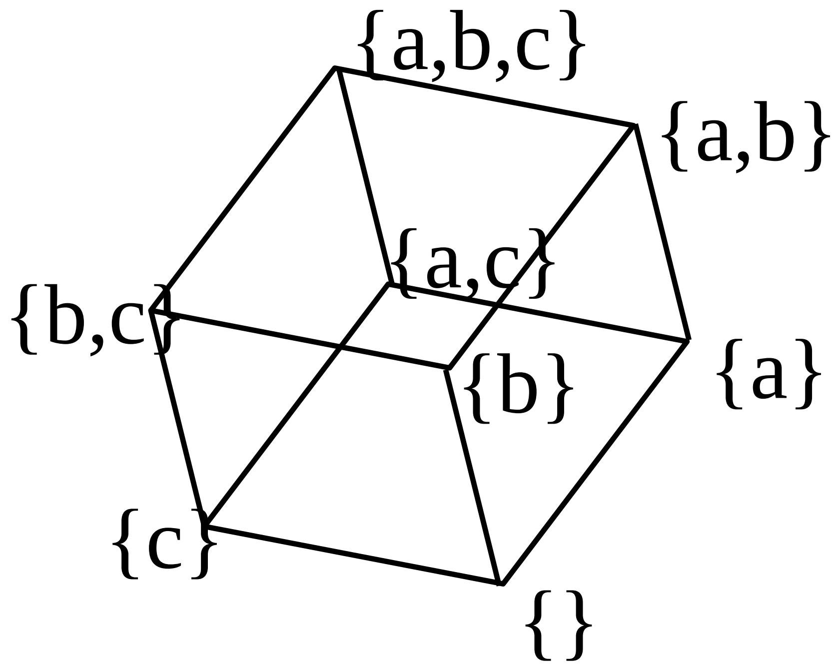
- Combine to form superset lattice for multi-element sets

Product Lattice

- (S, Λ) is product lattice of (S_1, Λ_1) and (S_2, Λ_2)
  - $S = S_1 \times S_2$ (domain)
  - For (a_1,a_2) and $(b_1, b_2) \in S$
    - $(a_1,a_2) \Lambda (b_1,b_2) = (a_1 \Lambda_1 b_1, a_2 \Lambda_2 b_2)$
    - $(a_1, a_2) \leq (b_1, b_2)$ iff $a_1 \leq_1 b_1$ and $a_2 \leq_2 b_2$
  - $\leq$ relation follows from Λ
- Product of lattices is associative
- $\Lambda_1, \Lambda_2, \ldots$ are called component lattices

Product Lattice





Height of a Semilattice

- Length of a chain $x1 \leq x2 \leq \ldots \leq xk$ is k
- K = max over length of all chains in the semilattice
- Height of semilattice = K-1

Data Flow Analysis Framework

- (D, S, $\wedge$, F)
- D: direction, Forward or Backward
- (S, $\wedge$): Semilattice Domain and meet
- F: family of transfer functions, S->S

Transfer Functions

- F: family of functions, S -> S. Includes
  - functions suitable for the boundary conditions (constant transfer functions for ENTRY and EXIT nodes)
  - Identity function I: $I(x) = x \ \ \forall x \in S$
- Closed under composition
  - $f, g \in F, h(x) = g\big(f(x)\big) \Rightarrow h \in F$

Monotonic Functions

- (S,≤) : a poset
- f: S->S is monotonic iff
  $\forall x, y \in S \quad x \leq y \Rightarrow f(x) \leq f(y)$
- Composition preserves monotonicity
  - If f and g are monotonic, h = f.g, then h is also monotonic

Monotone Frameworks

- (D, S, ∧, F) is monotone if the family F consists of monotonic functions only
  $f \in F, x, y \in S\ x \leq y \Rightarrow f(x) \leq f(y)$
- Equivalently
  $f \in F, x, y \in S\ f(x \wedge y) \leq f(x) \wedge f(y)$
  - Proof: Exercise

A Fixed Point Theorem

- $f: S -> S$ a monotonic function
- $(S, \wedge)$ is a finite height semilattice,
- T is top element
- $f^0(x) = x, f^{i+1}(x) = f\left(f^i(x)\right), i \geq 0$
- The greatest fixed point of f is $f^k(\mathrm{T})$ where $f^{k+1}(\mathrm{T}) = f^k(\mathrm{T})$

Fixed Point Algorithm

```
// monotonic f on a meet semilattice
x := ⊤ ;

while (x != f(x))     x := f(x);

return x;
```